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The p -semisimple property for some generalizations of BCI algebras and its applications

Abstract. This paper presents some generalizations of BCI algebras (the RM, tRM, *RM, RM**, *RM**, aRM**, *aRM**, BCH**, BZ, pre-BZ and pre-BCI algebras). We investigate the p -semisimple property for algebras mentioned above; give some examples and display various conditions equivalent to p -semisimplicity. Finally, we present a model of mereology without antisymmetry (NAM) which could represent a tRM algebra.

1. Introduction

The class of all tRM algebras contains BCI, BCK algebras and many others. BCK and BCI algebras had been widely investigated by many authors, but for the first time they were introduced in 1966 by Y. Imai and K. Iséki [5, 8], as algebras connected to certain kinds of logics. In 1983, Q. P. Hu and X. Li [4] defined BCH algebras, which contain BCK and BCI algebras; and in 1991, R. Ye [20] introduced the notion of BZ algebra. Recently, A. Iorgulescu [6] has defined fourteen new distinct generalizations of BCI algebras; in particular, *RM, RM**, pre-BZ and pre-BCI algebras, which are contained in the class of RM algebras (an RM algebra is an algebra $(A; \rightarrow, 1)$ of type $(2, 0)$ satisfying the identities: $x \rightarrow x = 1$ and $1 \rightarrow x = x$). In [10], T. Lei and C. Xi defined p -semisimple BCI algebras. These algebras have been extensively investigated in many papers [1, 3, 9, 13, 18, 21], etc.

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In this manuscript, we would like to investigate the p-semisimple property for the above mentioned algebras. Section 2 will be dedicated to various aspects of BCI, BCK algebras needed to understand the results of our paper. In Section 3, which is the core of this work, we will give several examples of p-semisimple proper RM algebras, *RM algebras, *RM** algebras, pre-BZ algebras, etc. Moreover, various characterizations of p-semisimple algebras will be given. In Section 4 we will review the foundations of non-antisymmetric mereology and represent it as a tRM algebra.

2. Algebraic preliminaries

Let $\mathcal{A} = (A; \rightarrow, 1)$ be an algebra of type $(2, 0)$. \mathcal{A} can satisfy the following properties [6]:

- (An) $x \rightarrow y = 1 = y \rightarrow x \implies x = y$,
- (B) $(y \rightarrow z) \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] = 1$,
- (BB) $(y \rightarrow z) \rightarrow [(z \rightarrow x) \rightarrow (y \rightarrow x)] = 1$,
- (D) $y \rightarrow [(y \rightarrow x) \rightarrow x] = 1$,
- (Ex) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
- (L) $x \rightarrow 1 = 1$,
- (M) $1 \rightarrow x = x$,
- (Re) $x \rightarrow x = 1$,
- (*) $y \rightarrow z = 1 \implies (x \rightarrow y) \rightarrow (x \rightarrow z) = 1$,
- (**) $y \rightarrow z = 1 \implies (z \rightarrow x) \rightarrow (y \rightarrow x) = 1$,
- (Tr) $x \rightarrow y = 1 = y \rightarrow z \implies x \rightarrow z = 1$.

LEMMA 1 ([6])

Let $\mathcal{A} = (A; \rightarrow, 1)$ be an algebra of type $(2, 0)$. Then the following statements hold:

- (i) (Re) + (Ex) + (An) imply (M),
- (ii) (M) + (B) imply (*) + (**),
- (iii) (M) + (*) imply (Tr),
- (iv) (M) + (**) imply (Tr),
- (v) (Re) + (Ex) imply (D),
- (vi) (Re) + (Ex) + (*) imply (B).

An algebra $\mathcal{A} = (A; \rightarrow, 1)$ is a *BCH algebra* if (Re), (Ex) and (An) are fulfilled, while a BCH algebra \mathcal{A} is a BCI algebra if (B) holds. A *BCK algebra* is a BCI algebra satisfying (L). Additionally, an algebra \mathcal{A} is an *RM algebra* if (Re) and (M) hold. By Lemma 1 (i), BCH, BCI and BCK algebras are particular cases of RM algebras.

We now recall definitions of types of algebras that we will be working with.

- A *tRM algebra* is an RM algebra satisfying (Tr).
- A **RM algebra* is an RM algebra satisfying (*).
- An *RM** algebra* is an RM algebra satisfying (**).
- A **RM** algebra* is an RM algebra satisfying (*), (**).
- An *aRM** algebra* is an RM** algebra satisfying (An).
- A **aRM** algebra* is a *RM** algebra satisfying (An).
- A *BCH** algebra* is a BCH algebra satisfying (**).
- A *pre-BZ algebra* is an RM algebra satisfying (B).
- A *BZ algebra* (also called a weak BCC algebra) is a pre-BZ algebra satisfying (An).
- A *pre-BCI algebra* is a pre-BZ algebra satisfying (Ex).

By Lemma 1 (iii) and (iv), *RM and RM** algebras are particular cases of tRM algebras. By Lemma 1 (ii), every pre-BZ algebra is a *RM** algebra.

Let us denote by **RM**, **tRM**, ***RM**, ***RM****, ..., **pre-BCI** the classes of RM, tRM, *RM, *RM**, ..., pre-BCI algebras, respectively. Figure 1 represents the relationship between different classes of the above mentioned algebras (the arrows indicate the proper inclusion relation).

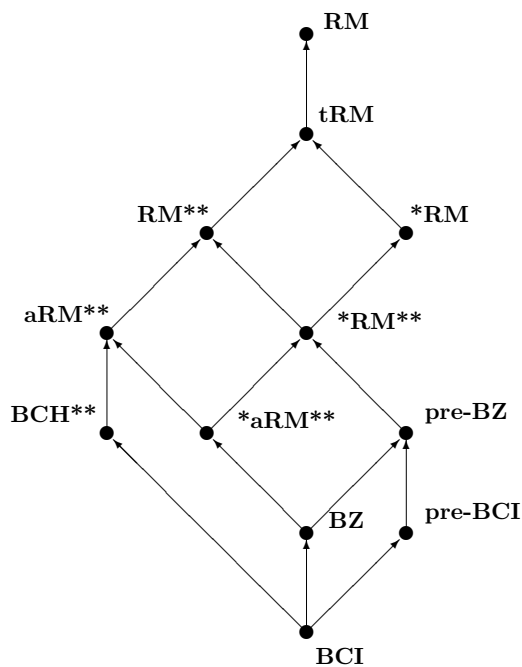


Fig. 1: Relationship between different classes of algebras

Now, let us define a binary relation “ \leq ” as follows

$$\forall x, y \in A \quad (x \leq y \iff x \rightarrow y = 1).$$

It can be verified that in BCK, BCI, BZ and BCH** algebras “ \leq ” is an order relation; instead in tRM, *RM, RM**, *RM**, pre-BZ and pre-BCI algebras, “ \leq ” is only reflexive and transitive, i.e. it is a pre-order relation.

Moreover, from [7] we recall that

- A *proper RM algebra* is an RM algebra such that (Ex), (An), (L), (B), (BB), (Tr), (*), (**) are not satisfied.
- A *proper tRM algebra* is a tRM algebra such that (Ex), (An), (L), (B), (BB), (*), (**) are not satisfied.
- A *proper *RM algebra* is a *RM algebra such that (Ex), (An), (L), (B), (BB), (**) are not satisfied.
- A *proper RM** algebra* is an RM** algebra such that (Ex), (An), (L), (B), (BB), (*) are not satisfied.
- A *proper *RM** algebra* is a *RM** algebra such that (Ex), (An), (L), (B), (BB) are not satisfied.
- A *proper aRM** algebra* is an aRM** algebra such that (Ex), (L), (B), (BB), (*) are not satisfied.
- A *proper *aRM** algebra* is a *aRM** algebra such that (Ex), (L), (B), (BB) are not satisfied.
- A *proper BCH** algebra* is a BCH** algebra such that (L), (B), (BB), (*) are not satisfied.
- A *proper pre-BZ algebra* is a pre-BZ algebra such that (BB), (Ex), (An), (L) are not satisfied.
- A *proper BZ algebra* is a BZ algebra such that (BB), (Ex), (L) are not satisfied.
- A *proper pre-BCI algebra* is a pre-BCI algebra such that (An), (L) are not satisfied.
- A *proper BCI algebra* is a BCI algebra such that (L) is not satisfied.

Let $\mathcal{A} = (A; \rightarrow, 1)$ be an algebra of type $(2, 0)$. We will write

$$x \approx y \iff x \leq y \wedge y \leq x \quad \text{for all } x, y \in A.$$

PROPOSITION 1

If \mathcal{A} is an RM algebra, then

$$x \approx 1 \iff x = 1$$

for all $x \in A$.

Proof. If $x \approx 1$, then $1 \leq x$, hence $1 = 1 \rightarrow x = x$ by (M). Obviously, $1 \approx 1$.

PROPOSITION 2

If \mathcal{A} is a tRM algebra, then “ \approx ” is an equivalence relation on \mathcal{A} .

Proof. Straightforward.

PROPOSITION 3

If \mathcal{A} is a *RM** algebra, then “ \approx ” is a congruence relation on \mathcal{A} .

Proof. By Proposition 2, “ \approx ” is an equivalence relation on \mathcal{A} . Let $x, y, a \in \mathcal{A}$ and $x \approx y$. By (*), $a \rightarrow x \approx a \rightarrow y$. Applying (**), we obtain $x \rightarrow a \approx y \rightarrow a$. Consequently, “ \approx ” is a congruence relation on \mathcal{A} .

LEMMA 2

If \mathcal{A} is a *RM** algebra and $x, y \in \mathcal{A}$, then

$$x/\approx \leq y/\approx \text{ in } \mathcal{A}/\approx \quad \text{if and only if} \quad x \leq y \text{ in } \mathcal{A}.$$

Proof. We have

$$\begin{aligned} x/\approx \leq y/\approx &\iff x/\approx \rightarrow y/\approx = 1/\approx \iff x \rightarrow y/\approx = 1/\approx \\ &\iff x \rightarrow y \approx 1 \iff x \rightarrow y = 1 \\ &\iff x \leq y. \end{aligned}$$

PROPOSITION 4

If \mathcal{A} is a *RM** algebra, then \mathcal{A}/\approx is a *aRM** algebra.

Proof. Obviously, \mathcal{A}/\approx verifies (Re) and (M). Let $x, y, z \in \mathcal{A}$ and $y/\approx \leq z/\approx$. By Lemma 2, we have $y \leq z$. Since (*) holds in \mathcal{A} , we obtain $x \rightarrow y \leq x \rightarrow z$. Therefore, $x \rightarrow y/\approx \leq x \rightarrow z/\approx$, that is, $x/\approx \rightarrow y/\approx \leq x/\approx \rightarrow z/\approx$. Thus, \mathcal{A}/\approx satisfies (*). Similarly, \mathcal{A}/\approx satisfies (**) and (An).

By Proposition 4, the following corollary follows.

COROLLARY 1

- (i) If \mathcal{A} is a pre-BZ algebra, then \mathcal{A}/\approx is a BZ algebra.
- (ii) If \mathcal{A} is a pre-BCI algebra, then \mathcal{A}/\approx is a BCI algebra.

3. The p-semisimple property

DEFINITION 1

An RM algebra \mathcal{A} is *p-semisimple* if, for each $x \in \mathcal{A}$,

$$(p-S) \quad x \leq 1 \implies x = 1.$$

EXAMPLE 1

Consider the set $A = \{a, b, c, 1\}$ and the operation “ \rightarrow ” given by the following table.

\rightarrow	a	b	c	1
a	1	1	b	a
b	1	1	c	b
c	1	b	1	c
1	a	b	c	1

We can observe that properties (Re) and (M) are satisfied; (An) is not satisfied for $(x, y) = (a, b)$; (B), (BB) and (Ex) are not satisfied for $(x, y, z) = (a, b, c)$; (L) is not satisfied for $x = a$; (*), (**), and (Tr) are not satisfied for $(x, y, z) = (c, a, b)$. Hence, $\mathcal{A} = (A; \rightarrow, 1)$ is a p-semisimple proper RM algebra.

EXAMPLE 2

Consider the set $A = \{a, b, c, 1\}$ with the following table of “ \rightarrow ”,

\rightarrow	a	b	c	1
a	1	1	b	a
b	1	1	c	b
c	c	b	1	c
1	a	b	c	1

Then the algebra $\mathcal{A} = (A; \rightarrow, 1)$ satisfies properties (Re), (M) and (Tr). It does not satisfy: (An) for a, b ; (B), (BB), (Ex), (*) and (**) for $(x, y, z) = (c, a, b)$; (L) for $x = a$. Thus \mathcal{A} is a p-semisimple proper tRM algebra.

EXAMPLE 3

Let $A = \{a, b, c, 1\}$ and “ \rightarrow ” be defined as follows

\rightarrow	a	b	c	1
a	1	1	b	a
b	1	1	c	b
c	b	b	1	c
1	a	b	c	1

It is easy to see that properties (Re), (M) and (*) (hence (Tr)) are satisfied; (An) is not satisfied for $(x, y) = (a, b)$; (B) and (BB) are not satisfied for $(x, y, z) = (b, a, c)$; (Ex) is not satisfied for $(x, y, z) = (a, c, a)$; (L) is not satisfied for $x = a$; (**) is not satisfied for $(x, y, z) = (c, a, b)$. Therefore, $\mathcal{A} = (A; \rightarrow, 1)$ is a p-semisimple proper *RM algebra.

EXAMPLE 4

Consider the set $A = \{a, b, c, 1\}$ with the following table of “ \rightarrow ”,

\rightarrow	a	b	c	1
a	1	1	b	c
b	1	1	b	c
c	b	b	1	c
1	a	b	c	1

Properties (Re), (M), (*) and (**) (hence (Tr)) are satisfied; (An) is not satisfied for $(x, y) = (a, b)$; (B) is not satisfied for $(x, y, z) = (b, c, b)$; (BB) is not satisfied for $(x, y, z) = (a, b, c)$; (Ex) is not satisfied for $(x, y, z) = (a, c, a)$; (L) is not satisfied for $x = a$. Thus $\mathcal{A} = (A; \rightarrow, 1)$ is a p-semisimple proper *RM** algebra.

EXAMPLE 5

Let $A = \{a, b, c, d, e, f, 1\}$ and let “ \rightarrow ” be defined by the following table

\rightarrow	a	b	c	d	e	f	1
a	1	1	e	f	c	d	a
b	1	1	e	f	c	d	a
c	f	f	1	e	d	a	c
d	e	e	f	1	a	c	d
e	c	c	d	a	1	e	f
f	d	d	a	c	f	1	e
1	a	b	c	d	e	f	1

Properties (Re), (M) and (B) (hence (*), (**), (Tr)) are satisfied; (An) is not satisfied for $(x, y) = (a, b)$; (BB) is not satisfied for $(x, y, z) = (d, f, c)$; (Ex) is not satisfied for $(x, y, z) = (a, c, b)$; (L) is not satisfied for $x = a$. Therefore, $\mathcal{A} = (A; \rightarrow, 1)$ is a p-semisimple proper pre-BZ algebra.

EXAMPLE 6

Consider the set $A = \{a, b, c, d, e, 1\}$ with the following table for “ \rightarrow ”,

\rightarrow	a	b	c	d	e	1
a	1	d	e	b	c	a
b	e	1	d	c	a	b
c	d	e	1	a	b	c
d	b	c	a	1	d	e
e	c	a	b	e	1	d
1	a	b	c	d	e	1

Properties (Re), (M), (B) (hence (*), (**), (Tr)) and (An) are satisfied; (Ex) and (BB) are not satisfied for $(x, y, z) = (d, b, a)$; (L) is not satisfied for $x = a$. Hence, $\mathcal{A} = (A; \rightarrow, 1)$ is a p-semisimple proper BZ algebra.

REMARK 1

Note that examples of p-semisimple proper RM**, *aRM** and pre-BCI algebras can be found in [7] (see Examples 9.4, 9.13 and 8.9).

EXAMPLE 7

Let $A = \{a, b, c, 1\}$ and “ \rightarrow ” be defined by the following table

\rightarrow	a	b	c	1
a	1	c	b	a
b	c	1	a	b
c	b	a	1	c
1	a	b	c	1

Properties (Re), (M), (B), (BB) (hence (*), (**), (Tr)), (Ex) and (An) are satisfied; (L) is not satisfied for $x = a$. Thus $\mathcal{A} = (A; \rightarrow, 1)$ is a p-semisimple proper BCI algebra.

Now, let us consider the following conditions.

- | | |
|---|--|
| (1) $x \leq y \implies x \approx y$. | (1') $x \leq y \implies x = y$. |
| (2) $(y \rightarrow x) \rightarrow x \approx y$. | (2') $(y \rightarrow x) \rightarrow x = y$. |
| (3) $(y \rightarrow 1) \rightarrow 1 \approx y$. | (3') $(y \rightarrow 1) \rightarrow 1 = y$. |
| (4) $a \rightarrow x \approx b \rightarrow x \implies a \approx b$. | (4') $a \rightarrow x = b \rightarrow x \implies a = b$. |
| (5) $a \rightarrow 1 \approx b \rightarrow 1 \implies a \approx b$. | (5') $a \rightarrow 1 = b \rightarrow 1 \implies a = b$. |
| (6) $x \rightarrow y \approx (y \rightarrow x) \rightarrow 1$. | (6') $x \rightarrow y = (y \rightarrow x) \rightarrow 1$. |
| (7) $x \rightarrow a \approx x \rightarrow b \implies a \approx b$. | (7') $x \rightarrow a = x \rightarrow b \implies a = b$. |
| (8) $(x \rightarrow 1) \rightarrow y \approx (y \rightarrow 1) \rightarrow x$. | (8') $(x \rightarrow 1) \rightarrow y = (y \rightarrow 1) \rightarrow x$. |

PROPOSITION 5

- (i) If \mathcal{A} is an RM^{**} algebra, then (p-S) \iff (1).
- (ii) If \mathcal{A} is an aRM^{**} algebra, then (p-S) \iff (1').

Proof. (i). Let \mathcal{A} be a p-semisimple RM^{**} algebra and suppose that $x \leq y$. By (**), $y \rightarrow x \leq x \rightarrow x = 1$. Hence, $y \rightarrow x = 1$, that is, $y \leq x$. Consequently, $x \approx y$.

Conversely, let (1) hold in \mathcal{A} . Assume that $x \leq 1$. Then $x \approx 1$, and by Proposition 1, $x = 1$. Therefore, \mathcal{A} is p-semisimple.

(ii). This proof is obvious from (i).

THEOREM 1

- (i) Let \mathcal{A} be an RM^{**} algebra satisfying (D). Then (p-S) \iff (k) for $k = 1, 2, 3, 4, 5$.
- (ii) Let \mathcal{A} be an aRM^{**} algebra with property (D). Then (p-S) \iff (k') for $k = 1, 2, 3, 4, 5$.

Proof. (i). From Proposition 5 it follows that (p-S) \iff (1).

(1) \implies (2). By (D), $y \leq (y \rightarrow x) \rightarrow x$. Applying (1), we obtain (2).

(2) \implies (3). Putting $x = 1$ in (2), we have (3).

(2) \implies (4). Let $a \rightarrow x \approx b \rightarrow x$. Since \mathcal{A} satisfies (**), we conclude that $(a \rightarrow x) \rightarrow x \approx (b \rightarrow x) \rightarrow x$. Applying (2), we get $a \approx b$.

(4) \implies (5). Putting $x = 1$ in (4), we have (5).

(3) \implies (p-S). Let $x \leq 1$. Then $x \approx (x \rightarrow 1) \rightarrow 1 = 1$, and $x = 1$.

(5) \implies (p-S). Let $x \leq 1$. Then $x \rightarrow 1 = 1 = 1 \rightarrow 1$. By (5), $x \approx 1$, and therefore $x = 1$.

Consequently, (p-S) \iff (k) for $k = 1, 2, 3, 4, 5$, and we obtain (i).

Observe that (ii) follows from (i).

THEOREM 2

(i) If \mathcal{A} is a pre-BZ algebra, then (p-S) \iff (k) for $k = 1, 3, 6$.

(ii) If \mathcal{A} is a BZ algebra, then (p-S) \iff (k') for $k = 1, 3, 6$.

Proof. (i). Since \mathcal{A} is an RM** algebra, (p-S) \iff (1).

(1) \implies (6). By (Re) and (B), $(y \rightarrow x) \rightarrow 1 = (y \rightarrow x) \rightarrow (y \rightarrow y) \geq x \rightarrow y$. Applying (1), we get (6).

(6) \implies (3). Putting $x = 1$ in (6), we have $(y \rightarrow 1) \rightarrow 1 \approx 1 \rightarrow y = y$.

(3) \implies (p-S). This follows from the proof of Theorem 1.

The proof of (i) is complete.

Notice that (ii) follows from (i).

A RM algebra is said to be a *BF algebra* ([19]) if it satisfies (6'). By Theorem 2 (ii) we obtain the following corollary.

COROLLARY 2

Every p-semisimple BZ algebra is a BF algebra.

THEOREM 3

(i) If \mathcal{A} is a pre-BZ algebra satisfying (D), then (p-S) \iff (k) for $k = 1, \dots, 7$.

(ii) If \mathcal{A} is a BZ algebra satisfying (D), then (p-S) \iff (k') for $k = 1, \dots, 7$.

Proof. (i). By Theorems 1 and 2, we deduce that (p-S) \iff (k) for $k = 1, \dots, 6$.

(p-S) \implies (7). Let $x \rightarrow a \approx x \rightarrow b$. Then $(x \rightarrow a) \rightarrow 1 \approx (x \rightarrow b) \rightarrow 1$. From (6) it follows that $a \rightarrow x \approx b \rightarrow x$. By (4), $a \approx b$.

(7) \implies (p-S). Let $x \leq 1$. We have $x \rightarrow 1 = 1 = x \rightarrow x$. Then $1 \approx x$, that is, $x = 1$. Consequently, \mathcal{A} is p-semisimple.

Thus (i) holds.

Observe that (ii) follows from (i).

THEOREM 4

If \mathcal{A} is a BCH** algebra, then (p-S) \iff (k') for $k = 1, \dots, 7$.

Proof. By Lemma 1 (v), \mathcal{A} satisfies (D). Applying Theorem 1 (ii), we conclude that (p-S) \iff (k') for $k = 1, 2, 3, 4, 5$.

(2') \implies (6'). By (Re), (Ex) and (2'), $(y \rightarrow x) \rightarrow 1 = (y \rightarrow x) \rightarrow (x \rightarrow x) = x \rightarrow ((y \rightarrow x) \rightarrow x) = x \rightarrow y$, that is, (6') holds.

(6') \implies (2'). We have

$$\begin{aligned} (y \rightarrow x) \rightarrow x &\stackrel{(6')}{=} (x \rightarrow (y \rightarrow x)) \rightarrow 1 \stackrel{(Ex)}{=} (y \rightarrow (x \rightarrow x)) \rightarrow 1 \\ &\stackrel{(Re)}{=} (y \rightarrow 1) \rightarrow 1 \stackrel{(6')}{=} 1 \rightarrow y \stackrel{(M)}{=} y. \end{aligned}$$

(6') \implies (7'). Let $x \rightarrow a = x \rightarrow b$. Hence $(x \rightarrow a) \rightarrow 1 = (x \rightarrow b) \rightarrow 1$. Therefore, $a \rightarrow x = b \rightarrow x$ by (6'). Then $(a \rightarrow x) \rightarrow x = (b \rightarrow x) \rightarrow x$. By (2'), $a = b$.

(6') \implies (p-S). Let $x \leq 1$. Then $x \rightarrow 1 = 1 = x \rightarrow x$. Applying (7'), we get $x = 1$.

Thus (p-S) \iff (k') for $k = 1, \dots, 7$.

THEOREM 5

Let \mathcal{A} be a pre-BCI algebra. Then (p-S) \iff (k) for $k = 1, \dots, 8$.

Proof. By Lemma 1 (v), \mathcal{A} satisfies (D). By Theorem 3 (i) it follows that (p-S) \iff (k) for $k = 1, \dots, 7$.

Now let (3) hold. We have

$$\begin{aligned} (y \rightarrow 1) \rightarrow x &\stackrel{(3)}{\approx} (y \rightarrow 1) \rightarrow ((x \rightarrow 1) \rightarrow 1) \stackrel{(Ex)}{=} (x \rightarrow 1) \rightarrow ((y \rightarrow 1) \rightarrow 1) \\ &\stackrel{(3)}{\approx} (x \rightarrow 1) \rightarrow y. \end{aligned}$$

Thus (8) holds, and we obtain that (3) \implies (8). The converse implication is obvious.

COROLLARY 3

In BCI algebras, the property of p -semisimplicity is equivalent with condition (k'), for $k = 1, \dots, 8$.

THEOREM 6

Let \mathcal{A} be an RM algebra. \mathcal{A} is a p -semisimple pre-BZ algebra if and only if \mathcal{A} satisfies

$$x \rightarrow y \approx (z \rightarrow x) \rightarrow (z \rightarrow y). \quad (\overline{B})$$

Proof. Let \mathcal{A} be a p -semisimple pre-BZ algebra. Applying (B), we have

$$x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y).$$

Hence, from (1) we get (\overline{B}) .

Conversely, suppose that (\overline{B}) holds in \mathcal{A} . Then \mathcal{A} satisfies (B), and hence \mathcal{A} is a pre-BZ algebra. Taking $z = y$ in (\overline{B}) , we obtain

$$x \rightarrow y \approx (y \rightarrow x) \rightarrow (y \rightarrow y) \stackrel{(Re)}{=} (y \rightarrow x) \rightarrow 1.$$

Therefore, (6) holds in \mathcal{A} . By the use of Theorem 2 (i), we conclude that \mathcal{A} is p -semisimple.

Theorem 6 implies

COROLLARY 4

Let \mathcal{A} be an RM algebra. \mathcal{A} is a p -semisimple BZ algebra if and only if

$$x \rightarrow y = (z \rightarrow x) \rightarrow (z \rightarrow y)$$

for all $x, y, z \in A$.

LEMMA 3

Every p -semisimple aRM** algebra satisfies (*).

Proof. Let \mathcal{A} be a p -semisimple aRM** algebra, and $x, y, z \in A$. Suppose that $y \leq z$, then by Proposition 5 $y = z$, and $x \rightarrow y = x \rightarrow z$. Thus (*) holds in \mathcal{A} .

COROLLARY 5

*p-semisimple aRM** algebras coincide with p-semisimple *aRM** algebras.*

PROPOSITION 6

*Let \mathcal{A} be an RM algebra. \mathcal{A} is a p-semisimple *aRM** algebra if and only if it satisfies (1').*

Proof. Let \mathcal{A} be an RM algebra satisfying (1'). Clearly, (An), (*) and (**) hold in \mathcal{A} . Therefore, \mathcal{A} is an *aRM** algebra. By Proposition 5 (ii), \mathcal{A} is p-semisimple. Thus \mathcal{A} is a p-semisimple *aRM** algebra. The converse is obvious.

THEOREM 7

Let \mathcal{A} be an RM algebra. The following statements are equivalent

- (a) *\mathcal{A} is a p-semisimple BCI algebra,*
- (b) *\mathcal{A} is a p-semisimple BCH** algebra,*
- (c) *\mathcal{A} satisfies (Ex) and (1').*

Proof. (a) \implies (b). Obvious.

(b) \implies (c). By definition, \mathcal{A} satisfies (Ex), hence, by Proposition 5 (ii) we conclude that \mathcal{A} also satisfies (1').

(c) \implies (a). It is easy to see that (An), (*) and (**) hold in \mathcal{A} . Applying Lemma 1 (vi), we deduce that \mathcal{A} satisfies (B), and therefore, \mathcal{A} is a BCI algebra. By the use of Proposition 5 (ii), \mathcal{A} is p-semisimple.

COROLLARY 6

*p-semisimple BCH** algebras coincide with p-semisimple BCI algebras.*

4. Non-antisymmetric mereology as a representation of p-semisimple algebras

4.1. Some remarks on classical mereology

The invention of mereology was the answer for the set theoretical paradox discovered by B. Russell in foundations of mathematics. Mereology is a set theory based on Leśniewski's concept of a set [11]. It is also called a collective set theory because a set is conceived as an aggregate in which all its parts are interrelated. Mereological objects are assumed to be real; therefore, it is a set theory created for physical objects.

In comparison to classical (Zermelo-Fraenkel) set theory, mereology opens a new perspective of conceiving sets, i.e. a set is conceived as a whole and only from the perspective of the whole do we then distinguish elements. Let $I_1 = [0, 1]$. Mereologically, it can be considered as a set composed of two halves: $S_1 = [0, 1/2]$, $S_2 = [1/2, 1]$, therefore $I_1 = \{S_1, S_2\}$ or as a set composed of four quarters: $I_2 = \{Q_1, Q_2, Q_3, Q_4\}$, where $Q_1 = [0, 1/4]$, $Q_2 = [1/4, 1/2]$, $Q_3 = [1/2, 3/4]$, $Q_4 = [3/4, 1]$. In the case of halves, this set has two elements: S_1, S_2 ; in the case of quarters, the given set has four elements: Q_1, Q_2, Q_3, Q_4 . Since in mereology the concept of a class is synonymous with the concept of sum, we have

$I_1 = I_2$, i.e. $\{S_1, S_2\} = \{Q_1, Q_2, Q_3, Q_4\}$, but in ZFC set theory $I_1 \neq I_2$, i.e. $\{S_1, S_2\} \neq \{Q_1, Q_2, Q_3, Q_4\}$. Thus, the novelty of mereology provides a new way of conceiving objects: there are wholes which can be divided into parts in different ways, as exemplified by the decay of a physical particle *meson* π [16].

The extensionality axiom provides a sufficient condition for two sets to be equal. *Mereological Extensionality Axiom* (MEA), which is slightly different from the ZFC extensionality principle, is expressed as follows

$$\forall z (z \subset x \iff z \subset y \implies x = y)^*.$$

It states that two objects having all the same proper subsets are equal. We can observe that in the example of intervals I_1, I_2 , MEA holds. Moreover, MEA is also conserved for ZFC sets; in fact, I_2 has different subsets than I_1 , therefore $I_1 \neq I_2$. This difference between the ZFC extensionality axiom and the mereological extensionality axiom follows from the fact that a mereological set is not uniquely determined by its elements. In [15] it was shown that the mereological extensionality axiom follows from the antisymmetry of the inclusion relation “ \subseteq ”. Hence, it is the antisymmetry property for the inclusion relation which is crucial for ZFC set theory and mereology.

Now, let us present a classical extensional mereology, formally[†]. Let M be a universe of objects and “ \subseteq ” – the binary relation of being an ingredient. It is assumed that the relation “ \subseteq ” partially orders M ,

$$\forall x \in M (x \subseteq x), \tag{M1}$$

$$\forall x, y \in M (x \subseteq y \wedge y \subseteq x \implies x = y), \tag{M2}$$

$$\forall x, y, z \in M (x \subseteq y \wedge y \subseteq z \implies x \subseteq z). \tag{M3}$$

With the use of the relation “ \subseteq ” we may define three auxiliary relations: the relation of overlapping – “ \circ ”, disjointedness – “ \wr ” and the relation “ \subset ”, defined on the $M \times M$ in the following way

$$\forall x, y \in M (x \circ y \stackrel{\text{df}}{\iff} \exists z \in M (z \subseteq x \wedge z \subseteq y)),$$

$$\forall x, y \in M (x \wr y \stackrel{\text{df}}{\iff} \neg x \circ y),$$

$$\forall x, y \in M (x \subset y \stackrel{\text{df}}{\iff} x \subseteq y \wedge x \neq y).$$

Apart from conditions (M1)–(M3), in classical Mereology the so-called *Strong Supplementation Principle* is assumed

$$\forall x, y \in M (x \not\subseteq y \implies \exists z \in M (z \subseteq x \wedge z \wr y)). \tag{M4}$$

*This axiom is defined for the relation of proper inclusion, and it is valid for complex objects, i.e. objects having at least two subsets. Moreover, we adopted the term ‘subset’ for a mereological part, and the inclusion relation for the mereological relation of ingredients, because both have the same features as their mereological correspondences, respectively.

[†]A complete presentation of various axiomatizations of mereology is presented in [15].

A characteristic relation in mereology is the relation of a sum – *Sum* corresponding to the notion of Leśniewski’s class

$$x \text{ Sum } X \stackrel{\text{df}}{=} \forall y \in X \ y \subseteq x \wedge \forall z \in M \ (z \subseteq x \implies \exists w \in X \ (w \circ z)).$$

Finally, in mereological structures, the existence of Sum is assumed

$$\forall \emptyset \neq Z \subseteq M \ \exists x \in M \ (x \text{ Sum } Z).^\ddagger \quad (\text{M5})$$

This set of five postulates (M1)–(M5) defines the fundamental model of extensional Mereology (GEM–*Ground Extensional Mereology*), which is also known as a **mereological structure**. In 1974 Clay and Loeb showed that GEM forms a model isomorphic to a Boolean algebra without a null element [2, 12].

4.2. Non-antisymmetric mereology

In [15] we analyzed a model of GEM without antisymmetry, which was called: a non-antisymmetric mereology–NAM. The relation of division could become a fundamental relation of NAM since it is a pre-ordering. Formally, let M be a space of physical objects, which are collective sets (classes), and “|” be the relation of ingredient fulfilling the following two axioms:

$$\forall x \in M \ x|x, \quad (\text{NAM1})$$

$$\forall x, y, z \in M \ (x|y \wedge y|z \implies x|z). \quad (\text{NAM3})$$

In NAM we assume the existence of objects that are mutually divisible by each other and different, i.e.

$$\exists x, y \in M \ (x|y \wedge y|x \wedge x \neq y). \quad (\text{NAM2})$$

In [14] it was shown that in the absence of antisymmetry, we obtain two different definitions of proper division:

$$x|_{p1}y \equiv x|y \wedge x \neq y, \quad (\text{PP1})$$

$$x|_{p2}y \equiv x|y \wedge y \not|x. \quad (\text{PP2})$$

To define an order relation, (PP2) should be applied, since “|_{p2}” is transitive, while “|_{p1}” is not. Within NAM, we can introduce an ordering by the use of “|_{p2}”,

$$x \preceq y \iff (x|_{p2}y \vee x = y). \quad (\text{OR})$$

In this way (M, \preceq) can form a model of GEM with additional two axioms analogous to (M4) and (M5) defined in terms of “ \preceq ”.

[‡]The assumption that $X \neq \emptyset$ is redundant, as we reach contradiction otherwise [17].

4.3. Algebraic representations of NAM

Let M be a nonvoid subset of \mathbb{Z} and “|” be the relation of divisibility on M . Clearly, “|” satisfies (NAM1) and (NAM3), that is, $(M; |)$ is a pre-ordered set. Moreover, if M contains elements a and b such that $a|b$, $b|a$ and $a \neq b$ (for example, 1 and -1 belong to M), then M also satisfies (NAM2).

EXAMPLE 8

Let $M = \mathbb{Z} - \{-1\}$ and “ \rightarrow ” be the operation on M defined as follows

$$x \rightarrow y = \begin{cases} 1, & \text{if } x = y \\ y, & \text{if } x|y \text{ and } x \neq y \\ x, & \text{if } x \not| y. \end{cases}$$

It is easy to see that $x \rightarrow x = 1$, $1 \rightarrow x = x$ and $x \leq y \iff x \rightarrow y = 1 \iff x = y$, for all $x, y \in M$. Hence $\mathcal{M} = (M; \rightarrow, 1)$ satisfies (Re), (M), (*), (**), (An) and it is p-semisimple. Properties (B), (BB) and (Ex) do not hold for $x = 4$, $y = 3$, $z = 2$. The property (L) does not hold for $x = 2$. Therefore, $(M; \rightarrow, 1)$ is a p-semisimple proper *aRM* algebra – an example of an algebra that is not considered in Section 3.

EXAMPLE 9

Let $(G; *, 1)$ be a group and “ \rightarrow ” will be defined on G as follows

$$x \rightarrow y = \begin{cases} 1, & \text{if } x = y \\ x * y, & \text{if } x \neq y. \end{cases}$$

Then $(G; \rightarrow, 1)$ is a p-semisimple tRM algebra. Properties (Re), (M), (Tr) are fulfilled; other properties depend on G .

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