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Ann. Univ. Paedagog. Crac. Stud. Math. 22 (2023), 13-20
DOI: 10.2478/aupcsm-2023-0002

FOLIA 385

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica 22 (2023)

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K-functional related to the Deformed Hankel Transform

Abstract. The main result of the paper is the proof of the equivalence theorem for a K-functional and a modulus of smoothness for the Deformed Hankel Transform. Before that, we introduce the K-functional associated to the Deformed Hankel Transform.

1. Introduction and Preliminaries

In [2], Belkina and Platonov established the equivalence theorem for a K-functional and a modulus of smoothness for the Dunkl transform in the Hilbert space $\mathbf{L}^2(\mathbb{R}, |x|^{2\alpha+1})$, $\alpha \geq \frac{-1}{2}$, using a Dunkl translation operator.

In this paper, we prove the generalization of this theorem for the Deformed Hankel transform $\mathcal{F}_\kappa$, with a parameter $\kappa > \frac{1}{4}$. For this purpose, we use the deformed Hankel translation operator, this result is analogous of the statement proved in ([1], [2], [8], [10], [11]).

We recapitulate some facts about harmonic analysis related to the deformed Hankel transform, consider the differential operator

$$\mathbb{L}_\kappa(.) = |.|\Lambda_\kappa(.),$$

where Λ_κ is the Dunkl Laplacian defined by $\Lambda_\kappa = \frac{d^2}{dx^2} + \frac{2\kappa}{x}\frac{d}{dx} - \frac{\kappa}{x^2}(1-S)$, where $Sf(x) := f(-x)$.

The deformed Hankel kernel $B_\kappa(\lambda x)$ is given, for $\kappa > \frac{1}{4}$, by

$$B_\kappa(\lambda x) = j_{2\kappa-1}(2\sqrt{|\lambda x|}) - \frac{\lambda x}{2\kappa(2\kappa+1)} j_{2\kappa+1}(2\sqrt{|\lambda x|}), \tag{1}$$

AMS (2020) Subject Classification: 41A36, 44A20.
Keywords and phrases: Deformed Hankel transform, generalized translation operator, Jackson's direct theorems, K-Functionals, Modulus of smoothness.
ISSN: 2081-545X, e-ISSN: 2300-133X.

where j_α denotes the normalized Bessel function of order α, defined by

$$j_\alpha(u) = 2^\alpha \Gamma(\alpha+1) u^{-\alpha} J_\alpha(u) = \Gamma(\alpha+1) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(\alpha+m+1)} \left(\frac{u}{2}\right)^{2m}. \qquad (2)$$

It satisfies the following differential-difference equation

$$|x|\Lambda_\kappa B_\kappa(\lambda x) = -|\lambda| B_\kappa(\lambda x).$$

From (1) and (2), one can easily find that B_κ has the properties

$$B_\kappa(0) = 1 \quad \text{and} \quad |B_\kappa(\lambda x)| \le 1 \text{ for all } \lambda, x \in \mathbb{R},$$

and from [9], we have

$$\lim_{\lambda x \to +\infty} B_\kappa(\lambda x) = 0. \qquad (3)$$

The deformed Hankel translation operator T_y^κ is defined by

$$T_y^\kappa f(x) = \int_{\mathbb{R}} f(z) K_\kappa(x,y,z) d\mu_\kappa(z),$$

where $d\mu_\kappa(x) = \frac{1}{2\Gamma(2\kappa)} |x|^{2\kappa-1} dx$ and for all $x, y \in \mathbb{R}^*$, the kernel K_κ is given by

$$K_\kappa(x,y,z) = 2\Gamma(2\kappa) W_{2\kappa-1}(\sqrt{|x|}, \sqrt{|y|}, \sqrt{|z|}) \nabla_\kappa(x,y,z),$$

where W_α is the positive Bessel kernel given by

$$W_\alpha(u,v,w) = \frac{\Gamma(\alpha+1)}{2^{2\alpha-1}\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})} \frac{\{[(u+v)^2 - w^2][w^2 - (u-v)^2]\}^{\alpha-\frac{1}{2}}}{(uvw)^{2\alpha}} \chi_{]|u-v|,u+v[}(w),$$

and

$$\begin{aligned}\nabla_\kappa(x,y,z) = &\frac{1}{4}\Big\{1 + \frac{\operatorname{sgn}(xy)}{4\kappa-1}[4\kappa\Delta(|x|,|y|,|z|)^2 - 1] \\ &+ \frac{\operatorname{sgn}(xz)}{4\kappa-1}[4\kappa\Delta(|z|,|x|,|y|)^2 - 1] \\ &+ \frac{\operatorname{sgn}(yz)}{4\kappa-1}[4\kappa\Delta(|z|,|y|,|x|)^2 - 1]\Big\}\end{aligned}$$

and $\Delta(u,v,w) = \frac{1}{2\sqrt{uv}}(u+v-w)$, $u, v, w \in \mathbb{R}_+^*$. We recall that $L^p(d\mu_\kappa), \kappa > \frac{1}{4}$, is the set of all measurable functions f on $\mathbb{R}$ satisfying

$$\|f\|_{p,\kappa} = \left(\int_{\mathbb{R}} |f(x)|^p d\mu_\kappa(x)\right)^{\frac{1}{p}} < \infty.$$

Let $f \in L^1(d\mu_\kappa)$, the deformed Hankel transform $\mathcal{F}_\kappa$ is defined by

$$\mathcal{F}_\kappa f(\lambda) = \int_{\mathbb{R}} f(x) B_\kappa(\lambda x) d\mu_\kappa(x), \qquad \lambda \in \mathbb{R}.$$

We have $\mathcal{F}_\kappa(f) \in \mathcal{C}_0(\mathbb{R})$. Moreover we have

$$\|\mathcal{F}_\kappa f\|_{\infty,\kappa} \leq \|f\|_{1,\kappa}.$$

It is well-known (see [3],[4],[5],[6]) that the deformed Hankel transform $\mathcal{F}_\kappa$ satisfies the following properties.

(a) Its inverse formula is given by

$$f(x) = \int_{\mathbb{R}} \mathcal{F}_\kappa f(\lambda) B_\kappa(\lambda x) d\mu_\kappa(\lambda).$$

(b) The Plancherel formula states

$$\|\mathcal{F}_\kappa f\|_{2,\kappa} = \|f\|_{2,\kappa}.$$

(c) From [9], we have

$$\mathcal{F}_\kappa(\mathbb{L}_\kappa^r f)(\lambda) = (-1)^r |\lambda|^r \mathcal{F}_\kappa f(\lambda), \qquad r \in \mathbb{N}, \tag{4}$$

where $\mathbb{L}_\kappa^r f = \mathbb{L}_\kappa(\mathbb{L}_\kappa^{r-1} f)$ and $\mathbb{L}_\kappa^0 f = f$.

(d) The generalized translation operator T_y^κ, verifies

$$\mathcal{F}_\kappa(T_y^\kappa f)(\lambda) = B_\kappa(\lambda y) \mathcal{F}_\kappa f(\lambda) \tag{5}$$

and we have $\|T_y^\kappa f\|_{\kappa,p} \leq A_\kappa \|f\|_{\kappa,\mathrm{p}}$ for all $1 \leq p \leq \infty$ and $y \in \mathbb{R}$.

Let $\mathcal{W}_{2,\kappa}^m$ be the Sobolev space constructed by the $\mathbb{L}_\kappa$ operator that is

$$\mathcal{W}_{2,\kappa}^m := \{f \in L^2(d\mu_\kappa) : \ \mathbb{L}_\kappa^j f \in L^2(d\mu_\kappa), \ j = 1, 2, \ldots, m\},$$

where $\mathbb{L}_\kappa^j f = \mathbb{L}_\kappa(\mathbb{L}_\kappa^{j-1} f)$ and $\mathbb{L}_\kappa^0 f = f$. Now we define the finite differences of order $m \in \mathbb{N}$ and step $h > 0$ by

$$\Delta_h^m f(\lambda) = (T_h^\kappa - I)^m f(\lambda),$$

where I denotes the unit operator.

Remark 1
For all $m \in \mathbb{N}$, we have

$$\Delta_h^m f(\lambda) = \sum_{0 \leq i \leq m} (-1)^{m-i} \binom{m}{i} (T_h^\kappa)^i f(\lambda).$$

Lemma 2
Let $f \in L^2(d\mu_\kappa)$, we have

$$\mathcal{F}_\kappa(\Delta_h^m f)(\lambda) = (B_\kappa(\lambda h) - 1)^m \mathcal{F}_\kappa(f)(\lambda). \tag{6}$$

Proof. On the basis of (5), we have

$$\mathcal{F}_\kappa\left(T_y^\kappa f\right)(\lambda) = B_\kappa(\lambda y)\mathcal{F}_\kappa f(\lambda),$$

then, by recurrence on i, we get

$$\mathcal{F}_\kappa((T_y^\kappa)^i f)(\lambda) = (B_\kappa(\lambda y))^i \mathcal{F}_\kappa f(\lambda),$$

hence

$$\begin{aligned}\mathcal{F}_\kappa(\Delta_h^m f)(\lambda) &= \sum_{0\le i\le m}(-1)^{m-i}\binom{m}{i}(B_\kappa(\lambda y))^i\mathcal{F}_\kappa f(\lambda),\\ &= \left(\sum_{0\le i\le m}(-1)^{m-i}\binom{m}{i}(B_\kappa(\lambda y))^i\right)\mathcal{F}_\kappa f(\lambda).\end{aligned}$$

Using Newton's formula, we obtain (6).

Definition 3
Let $f \in L^2(d\mu_\kappa)$ and $\delta > 0$. Then

(i) The generalized modulus of smoothness is defined by

$$w_m(f,\delta)_{2,\kappa} = \sup_{0<h\le\delta}\|\Delta_h^m f\|_{2,\kappa}.$$

(ii) The generalized K-functional is defined by

$$K_m(f,\delta)_{2,\kappa} = \inf\{\|f-g\|_{2,\kappa} + \delta\|\mathbb{L}_\kappa^m g\|_{2,\kappa} :\ g \in \mathcal{W}_{2,\kappa}^m\}.$$

The modulus of smoothness $w_m(f,\delta)_{2,\kappa}$ possesses the following properties (see, for example [7],[9]),

(a) $w_m(f,\delta)_{2,\kappa} \le A_\kappa 2^m\|f\|_{2,\kappa}$;

(b) if $f \in \mathcal{W}_{2,\kappa}^m$, then $w_m(f,\delta)_{2,\kappa} \le c(m,\kappa)\delta^m\left\|\mathbb{L}_\kappa^m f\right\|_{2,\kappa}$, where $c(m,\kappa)$ is a constant.

Throughout this paper, C denote a positive constant which may vary by line.

2. Main result

In order to prove Theorem 8 we need some preliminary results. The behaviour in 0 of the kernel $B_\kappa(\lambda x)$ could be deduced from [7] and [9], we get

$$B_\kappa(\lambda x) = 1 - \frac{1}{2\kappa}|\lambda x| - \frac{\lambda x}{2\kappa(2\kappa+1)} + \frac{\operatorname{sgn}(\lambda x)}{2\kappa(2\kappa+1)(2\kappa+2)}|\lambda x|^2 + o(|\lambda x|^2). \quad (7)$$

Lemma 4

(i) *There exist constants* $C > 0$ *and* $v > 0$ *such that if* $|\lambda x| \leq v$, *then*

$$|B_\kappa(\lambda x) - 1| \geq C|\lambda x|. \tag{8}$$

(ii) *There exist constants* $C > 0$ *and* $v > 0$ *such that if* $|\lambda x| \geq v$, *then*

$$|B_\kappa(\lambda x) - 1| \geq C. \tag{9}$$

Proof. (i). Using the relation (7), we obtain

$$\lim_{|\lambda x| \to 0} \frac{|B_\kappa(\lambda x) - 1|}{|\lambda x|} = \frac{1}{2\kappa} + \frac{\operatorname{sgn}(\lambda x)}{2\kappa(2\kappa+1)}.$$

This allows to get that there exist $C > 0$ and $v > 0$ such that

$$|\lambda x| \leq v \Longrightarrow |B_\kappa(\lambda x) - 1| \geq C|\lambda x|.$$

(ii). From [12], we have the asymptotic formula for the normalized Bessel function j_α, when $x \to +\infty$,

$$j_\alpha(x) = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})}\Big(\frac{2}{x}\Big)^{\alpha+\frac{1}{2}} \cos\Big(x - (2\alpha+1)\frac{\pi}{4}\Big) + o\Big(\frac{1}{x^{\frac{3}{2}}}\Big).$$

Therefore

$$\lim_{\lambda x \to +\infty} B_\kappa(\lambda x) = 0.$$

As a consequence there exists $v > 0$ such that if $|\lambda x| \geq v$ the inequality $|B_\kappa(\lambda x)| \leq \frac{1}{2}$ is true. We get the inequality

$$|B_\kappa(\lambda x) - 1| \geq C, \quad \text{where } C = \frac{1}{2}.$$

For any function $f \in L^2(d\mu_\kappa)$ and any number $v > 0$ we define the function

$$P_v(f)(x) := \int_{-v}^{v} \mathcal{F}_\kappa f(\lambda) B_\kappa(\lambda x) d\mu_\kappa(\lambda) = \mathcal{F}_\kappa^{-1}(\mathcal{F}_\kappa f(\lambda)\chi_v(\lambda)),$$

where

$$\chi_v(\lambda) = \begin{cases} 1, & \text{if } |\lambda| \leq v, \\ 0, & \text{if } |\lambda| > v. \end{cases}$$

$\mathcal{F}_\kappa^{-1}$ is the inverse deformed Hankel transform. One can easily prove that the function $P_v(f)$ is infinitely differentiable and belongs to all classes $\mathcal{W}^m_{2,\kappa}$, $m \in \mathbb{N}$.

Lemma 5

If $f \in L^2(d\mu_\kappa)$, *then*

$$\|f - P_v(f)\|_{2,\kappa} \leq C w_m(f, \delta)_{2,\kappa}, \qquad m \in \mathbb{N},$$

where $v > 0$ *and* $\delta > 0$.

Proof. Using the Plancherel identity, we have

$$\begin{aligned}\|f-P_v(f)\|_{2,\kappa}^2 &= \int_{\mathbb{R}} |1-\chi_v(\lambda)|^2 |\mathcal{F}_\kappa f(\lambda)|^2 d\mu_\kappa(\lambda)\\ &= \int_{|\lambda|>v} |\mathcal{F}_\kappa f(\lambda)|^2 d\mu_\kappa(\lambda).\end{aligned}$$

By (9), we have

$$|B_\kappa(\lambda h)-1| \geq C \qquad \text{for } |\lambda h| > v.$$

Therefore, from (6) and the Plancherel identity we deduce that

$$\begin{aligned}\|f-P_v(f)\|_{2,\kappa}^2 &\leq C^{-2m} \int_{|\lambda|>v} |B_\kappa(\lambda x)-1|^{2m} |\mathcal{F}_\kappa f(\lambda)|^2 d\mu_\kappa(\lambda)\\ &= C^{-2m} \int_{|\lambda|>v} |\mathcal{F}_\kappa((T_h^\kappa - I)^m f)(\lambda)|^2 d\mu_\kappa(\lambda)\\ &\leq C^{-2m} \int_{\mathbb{R}} |\mathcal{F}_\kappa((T_h^\kappa - I)^m f)(\lambda)|^2 d\mu_\kappa(\lambda)\\ &= C^{-2m} \|(T_h^\kappa - I)^m f\|_{2,\kappa}^2 = C^{-2m}\|\Delta_h^m f\|_{2,\kappa}^2.\end{aligned}$$

Hence

$$\|f-P_v(f)\|_{2,\kappa} \leq C^{-m}\|\Delta_h^m f\|_{2,\kappa} \leq C^{-m} w_m(f,\delta)_{2,\kappa},$$

the lemma is proved.

Lemma 6
For any $f \in L^2(d\mu_\kappa)$, we have

$$\|\mathbb{L}_\kappa^m(P_v(f))\|_{2,\kappa} \leq C|h|^{-m}\|\Delta_h^m f\|_{2,\kappa}, \qquad m \in \mathbb{N}, \tag{10}$$

where $v > 0$.

Proof. Relations (4), (6) and (8) together with the Plancherel identity yield

$$\begin{aligned}\|\mathbb{L}_\kappa^m(P_v(f))\|_{2,\kappa}^2 &= \int_{-v}^{v} |\lambda|^{2m} |\mathcal{F}_\kappa f(\lambda)|^2 d\mu_\kappa(\lambda)\\ &\leq C^{-2m}|h|^{-2m} \int_{-v}^{v} |B_\kappa(\lambda h)-1|^{2m} |\mathcal{F}_\kappa f(\lambda)|^2 d\mu_\kappa(\lambda)\\ &\leq C^{-2m}|h|^{-2m} \int_{\mathbb{R}} |\mathcal{F}_\kappa(\Delta_h^m f)(\lambda)|^2 d\mu_\kappa(\lambda)\\ &= C^{-2m}|h|^{-2m}\|\Delta_h^m f\|_{2,\kappa}^2.\end{aligned}$$

Hence

$$\|\mathbb{L}_\kappa^m(P_v(f))\|_{2,\kappa} \leq C^{-m}|h|^{-m}\|\Delta_h^m f\|_{2,\kappa}.$$

This proves (10).

Corollary 7
The inequality

$$\|\mathbb{L}_\kappa^m(P_v(f))\|_{2,\kappa} \leq C\delta^{-m} w_m(f,\delta)_{2,\kappa},$$

holds for any $f \in L^2(d\mu_\kappa)$, $m \in \mathbb{N}$, $v > 0$ *and* $\delta > 0$.

Theorem 8
There are two positive constants $c_1 = c(m,\kappa)$ *and* $c_2 = c(m,\kappa)$ *such that*

$$c_1 w_m(f,\delta)_{2,\kappa} \leq K_m(f,\delta^m)_{2,\kappa} \leq c_2 w_m(f,\delta)_{2,\kappa} \tag{11}$$

for all $f \in L^2(d\mu_\kappa)$ *and* $\delta > 0$.

Proof. To prove the left-hand inequality in (11) it is sufficient to show that

$$w_m(f,\delta)_{2,\kappa} \leq C K_m(f,\delta^m)_{2,\kappa}. \tag{12}$$

Let $g \in \mathcal{W}_{2,\kappa}^m$. Using the properties of the modulus of smoothness (see [9], Properties 4.2) we obtain

$$\begin{aligned} w_m(f,\delta)_{2,\kappa} &\leq w_m(f-g,\delta)_{2,\kappa} + w_m(g,\delta)_{2,\kappa} \\ &\leq A_\kappa 2^m \|f-g\|_{2,\kappa} + C(m,\kappa)\delta^m \|\mathbb{L}_\kappa^m g\|_{2,\kappa} \\ &\leq C(\|f-g\|_{2,\kappa} + \delta^m \|\mathbb{L}_\kappa^m g\|_{2,\kappa}), \end{aligned}$$

where $C = \max(A_\kappa 2^m, C(m,\kappa))$. Taking the infimum over all $g \in \mathcal{W}_{2,\kappa}^m$ we arrive at inequality (12).

Now we prove the right-hand inequality in (11). If $g = P_v(f)$ for $v > 0$, then it follows from the definition of $K_m(f,\delta)_{2,\kappa}$ that

$$K_m(f,\delta^m)_{2,\kappa} \leq \|f - P_v(f)\|_{2,\kappa} + \delta^m \|\mathbb{L}_\kappa^m P_v(f)\|_{2,\kappa}.$$

It follows from Lemma 5, and Corollary 7 that

$$K_m(f,\delta^m)_{2,\kappa} \leq 2C w_m(f,\delta)_{2,\kappa},$$

which proves the right-hand inequality in (11).

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Received: May 30, 2022; final version: February 25, 2023;
available online: April 3, 2023.