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On a multivalued second order differential problem with Jensen multifunction

Abstract. The aim of this paper is to present a generalization of the results published in [5] and [8] for continuous Jensen multifunctions. In particular, we study a second order differential problem for multifunctions with the Hukuhara derivative.

Throughout this paper all vector spaces are supposed to be real. Let $X$ be a vector space. We introduce the notations:

$$A + B := \{a + b : a \in A, b \in B\} \quad \text{and} \quad \lambda A := \{\lambda a : a \in A\}$$

for $A, B \subset X$ and $\lambda \in \mathbb{R}$.

A subset $K$ of $X$ is called a cone if $tK \subset K$ for all $t \in (0, +\infty)$. A cone is said to be convex if it is a convex set.

Let $X$ and $Y$ be two vector spaces and let $K \subset X$ be a convex cone. A set-valued function $F: K \to n(Y)$, where $n(Y)$ denotes the family of all nonempty subsets of $Y$, is called additive if

$$F(x + y) = F(x) + F(y) \quad \text{for } x, y \in K$$

and $F$ is Jensen if

$$F\left( \frac{x + y}{2} \right) = \frac{F(x) + F(y)}{2} \quad \text{for } x, y \in K. \quad (1)$$

From now on, we assume that $X$ is a normed vector space, $c(X)$ denotes the family of all compact members of $n(X)$ and $cc(X)$ stands for the family of all convex sets of $c(X)$.

Lemma 1 ([4], Theorem 5.6)

Let $K$ be a convex cone with zero in $X$ and $Y$ be a topological vector space. A set-valued function $F: K \to c(Y)$ satisfies the equation (1) if and only if there exist an additive multifunction $A_F: K \to cc(Y)$ and a set $G_F \in cc(Y)$ such that

$$F(x) = A_F(x) + G_F \quad \text{for } x \in K.$$

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The Hukuhara difference $A - B$ of $A, B \in cc(X)$ is a set $C \in cc(X)$ such that $A = B + C$. By Rådström’s Cancellation Lemma [9] it follows that if this difference exists, then it is unique.

For a multifunction $F : [a, b] \to cc(X)$ such that there exist the Hukuhara differences $F(t) - F(s)$ as $a \leq s \leq t \leq b$, the Hukuhara derivative at $t \in (a, b)$ is defined by the formula

$$DF(t) = \lim_{k \to 0^+} \frac{F(t + k) - F(t)}{k} = \lim_{k \to 0^+} \frac{F(t) - F(t - k)}{k},$$

whenever both these limits exist with respect to the Hausdorff distance $h$ (see [3]). Moreover,

$$DF(a) = \lim_{s \to a^+} \frac{F(s) - F(a)}{s - a}, \quad DF(b) = \lim_{s \to b^-} \frac{F(b) - F(s)}{b - s}.$$

Let $X$ be a Banach space and let $[a, b] \subset \mathbb{R}$. If a multifunction $F : [a, b] \to cc(X)$ is continuous, then there exists the Riemann integral of $F$ (see [3]). We need the following properties of the Riemann integral.

**Lemma 2** ([7], Lemma 10)
If $F : [a, b] \to cc(X)$ is continuous, then $H(t) = \int_a^t F(u) \, du$ for $a \leq t \leq b$ is continuous.

**Lemma 3** ([10], Lemma 4)
If $F : [a, b] \to cc(X)$ is continuous and $H(t) = \int_a^t F(u) \, du$, then $DH(t) = F(t)$ for $a \leq t \leq b$.

Let $(K, +)$ be a semigroup. A one-parameter family $\{F_t : t \geq 0\}$ of set-valued functions $F_t : K \to n(K)$ is said to be a cosine family if

$$F_0(x) = \{x\} \quad \text{for } x \in K$$

and

$$F_{t+s}(x) + F_{t-s}(x) = 2F_t(F_s(x)) := 2 \bigcup_{y \in F_s(x)} F_t(y)$$

for $x \in K$ and $0 \leq s \leq t$.

Let $X$ be a normed space. A cosine family is called regular if

$$\lim_{t \to 0^+} h(F_t(x), \{x\}) = 0.$$

**Example 1**
Let $K = [0, +\infty)$ and $F_t(x) = [x \cosh at, x \cosh bt]$, where $0 \leq a \leq b$. Then $\{F_t : t \geq 0\}$ is a regular cosine family of continuous additive multifunctions.

**Example 2**
Let $K = [0, +\infty)$ and $F_t(x) = [x, x \cosh t + \cosh t - 1]$. Then $\{F_t : t \geq 0\}$ is a regular cosine family of continuous Jensen multifunctions.
We say that a cosine family \( \{F_t : t \geq 0\} \) is differentiable if all multifunctions \( t \mapsto F_t(x) \) \( (x \in K) \) have the Hukuhara derivative on \([0, +\infty)\).

**Lemma 4** ([8], Theorem)

Let \( X \) be a Banach space and let \( K \) be a closed convex cone with a nonempty interior in \( X \). Suppose that \( \{A_t : t \geq 0\} \) is a regular cosine family of continuous additive set-valued functions \( A_t: K \to \text{cc}(K) \), \( x \in A_t(x) \) for all \( x \in K \), \( t \geq 0 \) and \( A_t \circ A_s = A_s \circ A_t \) for all \( s, t \geq 0 \). Then this cosine family is twice differentiable and

\[
DA_t(x)|_{t=0} = \{0\}, \quad D^2A_t(x) = A_t(A(x))
\]

for \( x \in K \), \( t \geq 0 \), where \( DA_t(x) \) denotes the Hukuhara derivative of \( A_t(x) \) with respect to \( t \) and \( A(x) \) is the second Hukuhara derivative of this multifunction at \( t = 0 \).

We would like to obtain a similar result to the above one for a cosine family of continuous Jensen multifunctions. For this purpose we remind some properties of such a family.

**Lemma 5** ([6], Theorem 3)

Let \( X \) be a Banach space and let \( K \) be a closed convex cone in \( X \) such that \( \text{int} \ K \neq \emptyset \). A one-parameter family \( \{F_t : t \geq 0\} \) is a regular cosine family of continuous Jensen multifunctions \( F_t: X \to \text{cc}(X) \) such that \( x \in F_t(x) \) for all \( x \in K \), \( t \geq 0 \) and \( F_t \circ F_s = F_s \circ F_t \) for all \( s, t \geq 0 \) if and only if there exist a regular cosine family \( \{A_t : t \geq 0\} \) of continuous additive multifunctions \( A_t: K \to \text{cc}(K) \) such that \( x \in A_t(x) \) for all \( x \in K \), \( t \geq 0 \), \( A_t \circ A_s = A_s \circ A_t \) for all \( s, t \geq 0 \) and a set \( D \in \text{cc}(K) \) with zero for which conditions

\[
A_{t+s}(D) + A_{t-s}(D) = 2A_t(A_s(D)) \quad \text{for} \quad 0 \leq s \leq t,
\]

\[
F_t(x) = A_t(x) + \int_0^t \left( \int_0^s A_u(D) \, du \right) \, ds \quad \text{for} \quad t \geq 0
\]

hold.

Using Lemmas 2, 3, 4 and 5 we obtain the following theorem.

**Theorem 1**

Let \( X \) be a Banach space and let \( K \) be a closed convex cone with a nonempty interior in \( X \). Suppose that \( \{F_t : t \geq 0\} \) is a regular cosine family of continuous Jensen set-valued functions \( F_t: K \to \text{cc}(K) \), \( x \in F_t(x) \) for all \( x \in K \), \( t \geq 0 \) and \( F_t \circ F_s = F_s \circ F_t \) for all \( s, t \geq 0 \). Then this cosine family is twice differentiable and

\[
DF_t(x)|_{t=0} = \{0\}, \quad D^2F_t(x) = A_t(A(x) + D)
\]

for \( x \in K \), \( t \geq 0 \), where \( DF_t(x) \) denotes the Hukuhara derivative of \( F_t(x) \) with respect to \( t \), \( D \in \text{cc}(K) \) with zero, \( A(x) = D^2A_t(x)|_{t=0} \), \( \{A_t : t \geq 0\} \) is a regular cosine family of continuous additive multifunctions (as in Lemma 5).
Let $K$ be a closed convex cone with a nonempty interior in $X$. We consider a continuous multifunction $\Phi: [0, +\infty) \times K \to cc(K)$ Jensen with respect to the second variable. According to Lemma 1 there exist multifunctions $A_\Phi: [0, +\infty) \times K \to cc(X)$ additive with respect to the second variable and $G_\Phi: [0, +\infty) \to cc(X)$ such that

$$\Phi(t, x) = A_\Phi(t, x) + G_\Phi(t) \quad \text{for } x \in K, \ t \in [0, +\infty).$$

Setting $x = 0$ in (2) we have

$$\Phi(t, 0) = G_\Phi(t) \in cc(K) \quad \text{for } t \in [0, +\infty).$$

Since $A_\Phi(t, x) + \frac{1}{n}G_\Phi(t) = \frac{1}{n}\Phi(t, nx) \subset K$ for all $n \in \mathbb{N}$ and the set $K$ is closed, $A_\Phi(t, x) \in cc(K)$ for $x \in K, \ t \in [0, +\infty)$. Moreover, multifunctions $A_\Phi, G_\Phi$ are continuous. Indeed, $t \mapsto G_\Phi(t) = \Phi(t, 0)$ is continuous. As $\Phi$ and $G_\Phi$ are continuous, the multifunction $A_\Phi$ is also continuous.

Theorem 1 is a motivation for studying existence and uniqueness of a solution $\Phi: [0, +\infty) \times K \to cc(K)$, which is Jensen with respect to the second variable, of the following differential problem

$$\begin{align*}
\Phi(0, x) &= \Psi(x), \\
D\Phi(t, x)|_{t=0} &= \{0\}, \\
D^2\Phi(t, x) &= A_\Phi(t, H(x)),
\end{align*}$$

where $H, \Psi: K \to cc(K)$ are given continuous Jensen set-valued functions, $D\Phi(t, x)$ denotes the Hukuhara derivative of $\Phi(t, x)$ with respect to $t$ and $A_\Phi$ is the additive, with respect to the second variable, part of $\Phi$.

**Definition 1**
A multifunction $\Phi: [0, +\infty) \times K \to cc(K)$ is said to be a solution of the problem (3) if it is continuous, twice differentiable with respect to $t$ and $\Phi$ satisfies (3) everywhere in $[0, +\infty) \times K$ and in $K$, respectively, where $H, \Psi: K \to cc(K)$ are two given continuous Jensen multifunctions.

With the problem (3), we associate the following equation

$$\Phi(t, x) = \Psi(x) + \int_0^t \left( \int_0^s A_\Phi(u, H(x)) \, du \right) \, ds$$

for $x \in K, \ t \in [0, +\infty)$, where $H, \Psi: K \to cc(K)$ are given continuous Jensen multifunctions and $A_\Phi$ is the additive, with respect to the second variable, part of $\Phi$.

**Definition 2**
Let $H, \Psi: K \to cc(K)$ be two continuous Jensen set-valued functions. A map $\Phi: [0, +\infty) \times K \to cc(K)$ is said to be a solution of (4) if it is continuous and satisfies (4) everywhere.
THEOREM 2
Let $K$ be a closed convex cone with a nonempty interior in a Banach space and let $H, \Psi : K \to cc(K)$ be two continuous Jensen multifunctions. Let $\Phi : [0, +\infty) \times K \to cc(K)$ be a given Jensen with respect to the second variable set-valued function. This $\Phi$ is a solution of the problem (3) if and only if it is a solution of (4).

The proof of Theorem 2 is the same as the proof of Theorem 1 in [5].

In the proof of the next theorem we use the following lemmas.

LEMMA 6 ([12], Theorem 3)
Let $X$ and $Y$ be two normed spaces and let $K$ be a convex cone in $X$. Suppose that $\{F_i : i \in I\}$ is a family of superadditive lower semicontinuous in $K$ and $Q_+-$homogeneous set-valued functions $F_i : K \to n(Y)$. If $K$ is of the second category in $K$ and $\bigcup_{i \in I} F_i(x) \in b(Y)$ for $x \in K$, then there exists a constant $M \in (0, +\infty)$ such that

$$\sup_{i \in I} \|F_i(x)\| \leq M\|x\| \quad \text{for } x \in K.$$ 

Let $K$ be a closed convex cone in $X$. Applying Lemma 6 we can define the norm $\|F\|$ of a continuous additive multifunction $F : K \to n(K)$ to be the smallest element of the set

$$\{M > 0 : \|F(x)\| \leq M\|x\|, x \in K\}.$$ 

LEMMA 7
Let $K$ be a closed convex cone with a nonempty interior in a Banach space and let $H, \Psi : K \to cc(K)$ be two continuous Jensen multifunctions. Assume that a continuous multifunction $A : [0, T] \times K \to cc(K)$ is additive with respect to the second variable. Then the multifunction

$$F(t, x) := \Psi(x) + \int_0^t \left( \int_0^s A(u, H(x)) \, du \right) \, ds, \quad (t, x) \in [0, T] \times K$$

is Jensen with respect to the second variable and continuous.

Proof. The proof is based upon ideas found in the proof of Theorem 2 in the paper [5]. According to the proof of Theorem 1 in [5] we have that the multifunction $u \mapsto A(u, H(x))$ is continuous for all $x \in K$. We see that every set $F(t, x)$ belongs to $cc(K)$ and $F$ is Jensen with respect to the second variable.

Next we show that $F$ is continuous. Let $x, y \in K$ and $0 \leq t_1 \leq t_2 \leq T$. The set

$$A([0, T], x) = \bigcup_{t \in [0, T]} A(t, x)$$

is compact (see [1], Ch. IV, p. 110, Theorem 3), so it is bounded. Therefore, by Lemma 6, there exists a positive constant $M_A$ such that

$$\|A(u, a)\| \leq M_A\|a\|$$

(6)
for \( u \in [0, T] \) and \( a \in K \). This implies that
\[
\|A(u, H(x))\| \leq M_A \|H(x)\|
\]
for \( u \in [0, T] \). Thus
\[
\left\| \int_{t_1}^{t_2} \left( \int_0^s A(u, H(x)) \, du \right) \, ds \right\| \leq \int_{t_1}^{t_2} \left( \int_0^s \|A(u, H(x))\| \, du \right) \, ds
\]
\[
\leq \int_{t_1}^{t_2} \left( \int_0^s M_A \|H(x)\| \, du \right) \, ds
\]
\[
= \frac{t_2^2 - t_1^2}{2} M_A \|H(x)\|.
\]

From Lemma 5 in [11] and (6) there exists a positive constant \( M_0 \) such that
\[
h(A(u, a), A(u, b)) \leq M_0 \|A(u, \cdot)\| \|a - b\| \leq M_0 M_A \|a - b\|
\]
for \( u \in [0, T] \) and \( a, b \in K \). Therefore,
\[
A(u, a) \subset A(u, b) + M_0 M_A \|a - b\| S
\]
for \( u \in [0, T] \) and \( a, b \in K \).

Let \( \varepsilon > 0 \) and \( a \in H(x) \). There exists \( b \in H(y) \) for which
\[
\|a - b\| < d(a, H(y)) + \frac{\varepsilon}{M_0 M_A}.
\]
This shows that for every \( a \in H(x) \) there exists \( b \in H(y) \) such that
\[
A(u, a) \subset A(u, b) + M_0 M_A d(a, H(y)) S + \varepsilon S
\]
\[
\subset A(u, H(y)) + M_0 M_A h(H(x), H(y)) S + \varepsilon S,
\]
thus
\[
A(u, H(x)) \subset A(u, H(y)) + M_0 M_A h(H(x), H(y)) S + \varepsilon S
\]
for \( u \in [0, T] \). Since \( \varepsilon > 0 \) and \( x, y \in K \) are arbitrary, we obtain
\[
h(A(u, H(x)), A(u, H(y))) \leq M_0 M_A h(H(x), H(y))
\]
Hence and by properties of the Riemann integral we have
\[
h \left( \int_0^t \left( \int_0^s A(u, H(x)) \, du \right) \, ds, \int_0^t \left( \int_0^s A(u, H(y)) \, du \right) \, ds \right)
\]
\[
\leq \int_0^t \left( \int_0^s h(A(u, H(x)), A(u, H(y))) \, du \right) \, ds
\]
\[
\leq \int_0^t \left( \int_0^s M_0 M_A h(H(x), H(y)) \, du \right) \, ds
\]
\[
= \frac{t^2}{2} M_0 M_A h(H(x), H(y)).
\]
By (5), (7) and (8) we get
\[
\begin{align*}
 h(F(t_1, x), F(t_2, y)) & \leq h(\Psi(x), \Psi(y)) \\
 & \quad + h \left( \int_0^{t_1} \left( \int_0^s A(u, H(x)) du \right) ds, \int_0^{t_1} \left( \int_0^s A(u, H(y)) du \right) ds \right) \\
 & \leq h(\Psi(x), \Psi(y)) \\
 & \quad + h \left( \int_0^{t_1} \left( \int_0^s A(u, H(x)) du \right) ds, \int_0^{t_1} \left( \int_0^s A(u, H(y)) du \right) ds \right) \\
 & \quad + h \left( \{0\}, \left( \int_{t_1}^{t_2} A(u, H(y)) du \right) ds \right) \\
 & \leq h(\Psi(x), \Psi(y)) + \frac{t_1^2}{2} M_0 M_A h(H(x), H(y)) + \frac{t_2^2 - t_1^2}{2} M_A \|H(y)\|.
\end{align*}
\]
This shows that \( F \) is a continuous set-valued function, because \( \Psi \) and \( H \) are continuous.

**Theorem 3**

Let \( K \) be a closed convex cone with a nonempty interior in a Banach space and let \( H, \Psi: K \to cc(K) \) be two continuous Jensen multifunctions. Then there exists exactly one solution, Jensen with respect to the second variable, of the problem (3).

**Proof.** Fix \( T > 0 \). Let \( E \) be the set of all continuous set-valued functions \( \Phi: [0, T] \times K \to cc(K) \) such that \( x \mapsto \Phi(t, x) \) are Jensen. As it was shown, for \( \Phi \in E \) there exist continuous multifunctions \( A_\Phi: [0, T] \times K \to cc(K) \) additive with respect to the second variable and \( G_\Phi: [0, T] \to cc(K) \) such that \( \Phi(t, x) = A_\Phi(t, x) + G_\Phi(t) \) for \( x \in K, t \in [0, T] \).

Let \( \Phi, \Pi \in E \) be given by
\[
\Phi(t, x) = A_\Phi(t, x) + G_\Phi(t) \quad \text{and} \quad \Pi(t, x) = A_\Pi(t, x) + G_\Pi(t)
\]
(9)
for \( (t, x) \in [0, T] \times K \), where \( A_\Phi, A_\Pi: [0, T] \times K \to cc(K) \) are additive with respect to the second variable and \( G_\Phi(t), G_\Pi(t) \in cc(K) \). We define a functional \( \rho \) in \( E \times E \) as follows
\[
\rho(\Phi, \Pi) = \sup \{ h(A_\Phi(t, B), A_\Pi(t, B)) + h(G_\Phi(t), G_\Pi(t)) : 0 \leq t \leq T, B \in cc(K), \|B\| \leq 1 \}.
\]
We see that sets
\[
A_i([0, T], x) = \bigcup_{t \in [0, T]} A_i(t, x), \quad x \in K,
\]
\[
G_i([0, T]) = \bigcup_{t \in [0, T]} G_i(t),
\]
where \( i \in \{ \Phi, \Pi \} \) are compact (see [1], Ch. IV, p. 110, Theorem 3), so they are bounded. By Lemma 6 there exist positive constants \( M_{A_\Phi} \) and \( M_{A_\Pi} \) such that

\[
\| A_\Phi(t, x) \| \leq M_{A_\Phi} \| x \|, \quad \| A_\Pi(t, x) \| \leq M_{A_\Pi} \| x \|
\]

for \( t \in [0, T] \) and \( x \in K \). We note that

\[
h(A_\Phi(t, B), A_\Pi(t, B)) + h(G_\Phi(t), G_\Pi(t)) \\
\leq \| A_\Phi(t, B) \| + \| A_\Pi(t, B) \| + \| G_\Phi(\cdot, t, \cdot) \| + \| G_\Pi(\cdot, t, \cdot) \| \\
\leq M_{A_\Phi} + M_{A_\Pi} + \| G_\Phi(\cdot, t) \| + \| G_\Pi(\cdot, t) \|
\]

for \( t \in [0, T] \) and \( B \in cc(K) \) such that \( \| B \| \leq 1 \). Thus

\[
\rho(\Phi, \Pi) < +\infty,
\]

so the functional \( \rho \) is finite. It is easy to verify that \( \rho \) is a metric in \( E \).

As the space \((cc(K), h)\) is a complete metric space (see [2]), \((E, \rho)\) is also a complete metric space.

We introduce the map \( \Gamma \) which associates with every \( \Phi \in E \) the set-valued function \( \Gamma \Phi \) defined by

\[
(\Gamma \Phi)(t, x) := \Psi(x) + \int_0^t \left( \int_0^s A_\Phi(u, H(x)) \, du \right) \, ds
\]

for \((t, x) \in [0, T] \times K\). We see that every set \((\Gamma \Phi)(t, x)\) belongs to \( cc(K) \). By Lemma 7 the multifunction \( \Gamma \Phi \) is Jensen with respect to the second variable and continuous. Therefore, \( \Gamma : E \to E \).

Now, we prove that \( \Gamma \) has exactly one fixed point. According to Lemma 1 we take the notations \( \Psi(x) = A_\Psi(x) + G_\Psi \) and \( H(x) = A_H(x) + G_H \), \( x \in K \), where \( A_\Psi, A_H : K \to cc(K) \) are additive and \( G_\Psi, G_H \in cc(K) \). Let \( \Phi, \Pi \in E \) be of the form (9) and let \((t, x) \in [0, T] \times K\). We observe that

\[
(\Gamma \Phi)(t, x) = \Psi(x) + \int_0^t \left( \int_0^s A_\Phi(u, H(x)) \, du \right) \, ds
\]

\[
= A_\Psi(x) + G_\Psi + \int_0^t \left( \int_0^s A_\Phi(u, A_H(x)) \, du \right) \, ds
\]

\[
+ \int_0^t \left( \int_0^s A_\Phi(u, G_H) \, du \right) \, ds,
\]

thus the additive part \( A_{\Gamma \Phi}(t, x) \) of \( \Gamma \Phi \) is equal to

\[
A_\Psi(x) + \int_0^t \left( \int_0^s A_\Phi(u, A_H(x)) \, du \right) \, ds
\]
and similarly

\[ A_{\Gamma\Pi}(t, x) = A_{\Psi}(x) + \int_0^t \left( \int_0^s A_{\Pi}(u, A_H(x)) \, du \right) \, ds. \]

Hence and by properties of the Hausdorff metric we have

\[
\begin{align*}
& h(A_{\Gamma\Phi}(t, x), A_{\Gamma\Pi}(t, x)) + h(G_{\Gamma\Phi}(t), G_{\Gamma\Pi}(t)) \\
& = h\left( \int_0^t \left( \int_0^s A_{\Phi}(u, A_H(x)) \, du \right) \, ds, \int_0^t \left( \int_0^s A_{\Pi}(u, A_H(x)) \, du \right) \, ds \right) \\
& \quad + h\left( \int_0^t \left( \int_0^s A_{\Phi}(u, G_H) \, du \right) \, ds, \int_0^t \left( \int_0^s A_{\Pi}(u, G_H) \, du \right) \, ds \right) \\
& \leq \frac{t^2}{2!} \rho(\Phi, \Pi) \|A_H(x)\| + \frac{t^2}{2!} \rho(\Phi, \Pi) \|G_H\| \\
& \leq \frac{t^2}{2!} \rho(\Phi, \Pi) \max\{\|A_H(x)\|, \|G_H\|\}.
\end{align*}
\]

Suppose that

\[
h(A_{\Gamma^n\Phi}(t, x), A_{\Gamma^n\Pi}(t, x)) + h(G_{\Gamma^n\Phi}(t), G_{\Gamma^n\Pi}(t)) \leq \frac{t^{2n}}{(2n)!} \rho(\Phi, \Pi) \max\{\|A_H(x)\|, \|G_H\|\}^n \tag{10}
\]

for some \( n \in \mathbb{N} \). Then

\[
\begin{align*}
h(A_{\Gamma^{n+1}\Phi}(t, x), A_{\Gamma^{n+1}\Pi}(t, x)) + h(G_{\Gamma^{n+1}\Phi}(t), G_{\Gamma^{n+1}\Pi}(t)) \\
& = h\left( \int_0^t \left( \int_0^s A_{\Gamma^n\Phi}(u, A_H(x)) \, du \right) \, ds, \int_0^t \left( \int_0^s A_{\Gamma^n\Pi}(u, A_H(x)) \, du \right) \, ds \right) \\
& \quad + h\left( \int_0^t \left( \int_0^s A_{\Gamma^n\Phi}(u, G_H) \, du \right) \, ds, \int_0^t \left( \int_0^s A_{\Gamma^n\Pi}(u, G_H) \, du \right) \, ds \right) \\
& \leq \int_0^t \left( \int_0^s \frac{u^{2n}}{(2n)!} \rho(\Phi, \Pi) \max\{\|A_H(x)\|, \|G_H\|\}^{n+1} \, du \right) \, ds \\
& = \frac{t^{2n+2}}{(2n+2)!} \rho(\Phi, \Pi) \max\{\|A_H(x)\|, \|G_H\|\}^{n+1}.
\end{align*}
\]

This shows that (10) holds for all \( n \in \mathbb{N} \). Therefore,

\[
\rho(\Gamma^n\Phi, \Gamma^n\Pi) \leq \frac{2(T^2 \max\{\|A_H\|, \|G_H\|\})^n}{(2n)!} \rho(\Phi, \Pi), \quad n \in \mathbb{N}.
\]
We observe that for every \( T > 0 \) there exists \( n \in \mathbb{N} \) such that
\[
2^\left( T^2 \max\{\|A_H\|, \|G_H\|\} \right)^n < 1.
\]

By Banach Fixed Point Theorem we get that \( \Gamma^n \) has exactly one fixed point, whence it follows that \( \Gamma \) has exactly one fixed point. This means that there exists exactly one solution of the problem (3) for \((t, x) \in [0, T] \times K\).

Now we give an application. Let \( K \) be a closed convex cone with a nonempty interior in a Banach space. Suppose that \( \{F_t : t \geq 0\} \) and \( \{G_t : t \geq 0\} \) are regular cosine families of continuous Jensen multifunctions \( F_t : K \to \text{cc}(K), G_t : K \to \text{cc}(K) \) such that \( x \in F_t(x), \ x \in G_t(x), F_t \circ F_s = F_s \circ F_t, G_t \circ G_s = G_s \circ G_t \) for \( x \in K, s, t \geq 0 \)

and
\[ H(x) := D^2 F_t(x)|_{t=0} = D^2 G_t(x)|_{t=0}. \]

Then multifunctions \((t, x) \mapsto F_t(x)\) and \((t, x) \mapsto G_t(x)\) are Jensen with respect to \( x \) and satisfy (3) with \( \Psi(x) = \{x\} \). According to Theorem 3 we have \( F_t(x) = G_t(x) \) for \((t, x) \in [0, +\infty) \times K\). This means that if two regular cosine family as above have the same second order infinitesimal generator, then there are equal.

References

On a multivalued second order differential problem with Jensen multifunction

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