On some equations stemming from quadrature rules

Abstract. We deal with functional equations of the type
\[ F(y) - F(x) = (y - x) \sum_{k=1}^{n} f_k ((1 - \lambda_k)x + \lambda_k y), \]
connected to quadrature rules and, in particular, we find the solutions of the following functional equation
\[ f(x) - f(y) = (x - y)[g(x) + h(x + 2y) + h(2x + y) + g(y)]. \]
We also present a solution of the Stamate type equation
\[ yf(x) - xf(y) = (x - y)[g(x) + h(x + 2y) + h(2x + y) + g(y)]. \]
All results are valid for functions acting on integral domains.

1. Introduction

We deal with some equations connected to quadrature rules. Having a function \( f: \mathbb{R} \to \mathbb{R} \) we may approximate its integral using the following expression
\[ F(y) - F(x) \approx (y - x) \sum_{k=1}^{n} \alpha_k f((1 - \lambda_k)x + \lambda_k y) \]
(where \( F \) is a primitive function for \( f \)), which is satisfied exactly for polynomials of certain degree. One of the simplest functional equations connected to quadrature rules is an equation stemming from Simpson’s rule
\[ F(y) - F(x) = (y - x) \left[ \frac{1}{6} f(x) + \frac{2}{3} f \left( \frac{x + y}{2} \right) + \frac{1}{6} f(y) \right]. \]
Another example is given by the equation
\[ F(y) - F(x) = (y - x) \left[ \frac{1}{8} f(x) + \frac{3}{8} f \left( \frac{x + 2y}{3} \right) + \frac{1}{8} f \left( \frac{2x + y}{3} \right) + \frac{1}{8} f(y) \right]. \]
which is satisfied by polynomials of degree not greater than 3. The generalized version of this equation

\[ g(x) - f(y) = (x - y)[h(x) + k(sx + ty) + k(tx + sy) + h(y)] \]  

was considered during the 44th ISFE held in Louisville, Kentucky, USA by P.K. Sahoo [7]. The solution has been given in the class of functions \( f, g, h, k \) mapping \( \mathbb{R} \) into \( \mathbb{R} \) and such that \( g \) and \( f \) are twice differentiable, and \( k \) is four times differentiable.

On the other hand, M. Sablik [5] during the 7th Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities presented the general solution of this equation in the case \( s, t \in \mathbb{Q} \) without any regularity assumptions concerning the functions considered.

We deal with a special case of (1) (with \( s = 1, t = 2 \)) for functions acting on integral domains. However, it is easy to observe that if we take \( x = y \) in (1), then we immediately obtain that \( f = g \). Thus we shall find the solutions of the following functional equation

\[ f(x) - f(y) = (x - y)[g(x) + h(x + 2y) + h(2x + y) + g(y)]. \]  

Using the obtained result we will also present a solution of a similar Stamate type equation

\[ yf(x) - xf(y) = (x - y)[g(x) + h(2x + y) + h(x + 2y) + g(y)]. \]  

In the proof of Lemma 1 below we use the lemma established by M. Sablik [6] and improved by I. Pawlikowska [3]. First we need some notations. Let \( G, H \) be Abelian groups and \( SA^0(G, H) := H, SA^1(G, H) := \text{Hom}(G, H) \) (i.e., the group of all homomorphisms from \( G \) into \( H \)), and for \( i \in \mathbb{N} \), \( i \geq 2 \), let \( SA^i(G, H) \) be the group of all \( i \)-additive and symmetric mappings from \( G^i \) into \( H \). Furthermore, let \( \mathcal{P} := \{ (\alpha, \beta) \in \text{Hom}(G, G)^2 : \alpha(G) \subset \beta(G) \} \). Finally, for \( x \in G \) let \( x^i = (x, \ldots, x) \), \( i \in \mathbb{N} \).

**LEMMA 1**

Fix \( N \in \mathbb{N} \cup \{0\} \) and let \( I_0, \ldots, I_N \) be finite subsets of \( \mathcal{P} \). Suppose that \( H \) is uniquely divisible by \( N! \) and let the functions \( \varphi : G \rightarrow SA^i(G, H) \) and \( \psi_{i,(\alpha, \beta)} : G \rightarrow SA^i(G, H) ( (\alpha, \beta) \in I_i, i = 0, \ldots, N) \) satisfy

\[ \varphi_N(x)(y^N) + \sum_{i=0}^{N-1} \varphi_i(x)(y^i) = \sum_{i=0}^{N} \sum_{(\alpha, \beta) \in I_i} \psi_{i,(\alpha, \beta)}(\alpha(x) + \beta(y))(y^i) \]

for every \( x, y \in G \). Then \( \varphi_N \) is a polynomial function of order at most \( k - 1 \), where

\[ k = \sum_{i=0}^{N} \text{card} \left( \bigcup_{s=i}^{N} I_s \right). \]
Now we will state a simplified version of this lemma. We take $N = 1$ and we consider functions acting on an integral domain $P$. Moreover, we consider only homomorphisms of the type $x \mapsto yx$, where $y \in P$ is fixed.

**Lemma 2**

Let $P$ be an integral domain and let $I_0$, $I_1$ be finite subsets of $P^2$ such that for all $(a, b) \in I$, the ring $P$ is divisible by $b$. Let $\varphi_i, \psi_{i,(a, b)} : P \to P$ satisfy

$$
\varphi_1(x)y + \varphi_0(x) = \sum_{(a, b) \in I_0} \psi_{0,(a, b)}(ax + by) + y \sum_{(a, b) \in I_1} \psi_{1,(a, b)}(ax + by)
$$

for all $x, y \in P$. Then $\varphi_1$ is a polynomial function of order at most equal to \(\text{card}(I_0 \cup I_1) + \text{card} I_1 - 1\).

In the above lemmas a **polynomial function of order** $n$ means a solution of the functional equation $\Delta_h^{n+1} f(x) = 0$, where $\Delta_h^n$ stands for the $n$-th iterate of the difference operator $\Delta_h f(x) = f(x + h) - f(x)$. Observe that a continuous polynomial function of order $n$ is a polynomial of degree at most $n$ (see [2, Theorem 4, p. 398]).

It is also well known that if $P$ is an integral domain uniquely divisible by $n!$ and $f : P \to P$ is a polynomial function of order $n$, then

$$
f(x) = c_0 + c_1(x) + \ldots + c_n(x), \quad x \in P,
$$

where $c_0 \in P$ is a constant and

$$
c_i(x) = C_i(x, x, \ldots, x), \quad x \in P
$$

for some $i$-additive and symmetric function $C_i : P^i \to P$.

### 2. Results

We begin with the following lemma which will be useful in the proof of the main result. However, we state it a bit more generally.

**Lemma 3**

Let $P$ be an integral domain and let $f, f_k : P \to P$, $k = 0, \ldots, n$, be functions satisfying the equation

$$
f(y) - f(x) = (y - x) \sum_{k=0}^{n} f_k(a_k x + b_k y), \quad \text{ (4)}
$$

where $a_k, b_k \in P$ are given numbers such that for every $k \in \{0, \ldots, n\}$ we have $a_k \neq 0$ or $b_k \neq 0$.

Let $i \in \{0, \ldots, n\}$ be fixed. If $P$ is divisible by $a_i$, $b_i$ and also by $a_i b_k - a_k b_i$, $k = 0, \ldots, n; k \neq i$, then the function

$$
\tilde{f}(x) := (a_i + b_i) f_i((a_i + b_i)x)
$$

is a polynomial function of degree at most $2n + 1$. 
Moreover, if there exists \( k_1 \in \{0, 1, \ldots, n\} \) such that \( a_{k_1} = 0 \) or \( b_{k_1} = 0 \), then function \( f \) is a polynomial function of order at most \( 2n \) and if there exist \( k_1, k_2 \in \{0, \ldots, n\} \) such that \( a_{k_1} = b_{k_2} = 0 \), then \( \tilde{f} \) is a polynomial function of order at most \( 2n - 1 \).

Proof. Fix an \( i \in \{0, \ldots, n\} \), put in (4) \( x - b_i y \) and \( x + a_i y \) instead of \( x \) and \( y \), respectively, to obtain
\[
f(x + a_i y) - f(x - b_i y) = (a_i + b_i) y [f_0((a_0 + b_0)x + (a_i b_0 - a_0 b_i)y) + \ldots + f_i((a_i + b_i)x) + \ldots + f_n((a_n + b_n)x + (a_i b_n - a_n b_i)y)].
\]
(5)

There are two possibilities:
1. \( a_i, b_i \neq 0 \),
2. \( a_i = 0 \) or \( b_i = 0 \).

Let us consider the first case. Then from (5) we obtain
\[
y(a_i + b_i) f_i((a_i + b_i)x) = f(x + a_i y) - f(x - b_i y) - (a_i + b_i) y \sum_{k=0, k \neq i} f_k((a_k + b_k)x + (a_i b_k - a_k b_i)y),
\]
which means that
\[
y \tilde{f}(x) = f(x + a_i y) - f(x - b_i y) - (a_i + b_i) y \sum_{k=0, k \neq i} f_k((a_k + b_k)x + (a_i b_k - a_k b_i)y).
\]
(6)

Now we are in position to use Lemma 2 with
\[
I_0 = \{(1, -b_i), (1, a_i)\}
\]
and
\[
I_1 = \{(a_k + b_k, a_i b_k - a_k b_i) : k = 0, \ldots, n; k \neq i\}.
\]
We clearly obtain that \( \tilde{f} \) is a polynomial function of order at most equal to
\[
\text{card}(I_0 \cup I_1) + \text{card} I_1 - 1 \leq (n + 2) + n - 1 = 2n + 1.
\]

Further, if for example \( a_{k_1} = 0 \) for some \( k_1 \in \{0, \ldots, n\} \), \( k_1 \neq i \), then we have a summand
\[
f_{k_1}(b_{k_1} x + a_i b_{k_1} y) = f_{k_1}(b_{k_1} (x + a_i y))
\]
on the right-hand side of (6). Thus we put \( \tilde{f}_{k_1}(x) := f_{k_1}(b_{k_1} x) \) and (6) takes form
\[
y \tilde{f}(x) = f(x - b_i y) - f(x + a_i y)
\]
\[
- (a_i + b_i) y \left[ \sum_{k=0, k \neq i, k_1} f_k((a_k + b_k)x + (a_i b_k - a_k b_i)y) + \tilde{f}_{k_1}(x + a_i y) \right].
\]
Similarly as before we take
\[ I_0 = \{(1, -b_i), (1, a_i)\} \]
and
\[ I_1 = \{(a_k + b_k, a_i b_k - a_k b_i) : k = 0, \ldots, n; k \neq i, k_1\} \cup \{(1, a_i)\}. \]
In this case we have \( I_0 \cap I_1 = \{(1, a_i)\}, \) i.e.,
\[ \text{card}(I_0 \cup I_1) + \text{card} I_1 - 1 \leq (n + 1) + n - 1 = 2n. \]

The proof in the case \( a_{k_1} = b_{k_2} = 0 \) is similar.
Now we consider the case \( a_i = 0 \) or \( b_i = 0 \). Let for example \( a_i = 0 \), then from (6) we have
\[ y(b_i) f_i(b_i x) - f(x) = -f(x - b_i y) - b_i y \sum_{k=0,k \neq i}^n f_k((a_k + b_k)x - a_k b_i y), \]
i.e.,
\[ y b_i \tilde{f}(x) - f(x) = -f(x - b_i y) - b_i y \sum_{k=0,k \neq i}^n f_k((a_k + b_k)x - a_k b_i y). \]
In this case we take
\[ I_0 = \{(1, -b_i)\} \]
and
\[ I_1 = \{(a_k + b_k, -a_k b_i) : k = 0, \ldots, n; k \neq i\}. \]
Thus similarly as before \( \tilde{f} \) is a polynomial function of degree not greater than
\[ \text{card}(I_0 \cup I_1) + \text{card} I_1 - 1 \leq (n + 1) + n - 1 = 2n. \]

It is easy to see that if for some \( k_2 \in \{0, \ldots, n\}, b_{k_2} = 0 \), then \( \tilde{f} \) is a polynomial function of order at most \( 2n - 1 \).

Now we are in position to state the most important result of this paper. Namely, we give a general solution of (2) for functions acting on integral domains satisfying some assumptions.

**Theorem 1**

*Let \( P \) be an integral domain with unit element \( \mathbb{1} \), uniquely divisible by 5! and such that for every \( n \in \mathbb{N} \) we have \( n\mathbb{1} \neq 0 \). The functions \( f, g, h : P \to P \) satisfy the equation (2) if and only if there exist \( a, b, c, d, \bar{d}, \bar{e} \in P \) and an additive function \( A : P \to P \) such that*

\[ f(x) = 18ax^4 + 8bx^3 + cx^2 + 2dx + e, \quad x \in P, \]
\[ g(x) = 9ax^3 + 3bx^2 + cx - 3A(x) + d - \bar{d}, \quad x \in P, \]
\[ h(x) = ax^3 + bx^2 + A(x) + \bar{d}, \quad x \in P. \]
Proof. Assume that \( f, g, h : P \rightarrow P \) satisfy the equation (2). From Lemma 3 we know that \( g \) and \( h \) are polynomial functions of order at most 5. Therefore

\[
g(x) = c_0 + c_1(x) + c_2(x) + c_3(x) + c_4(x) + c_5(x), \quad x \in P
\]

and

\[
h(x) = d_0 + d_1(x) + d_2(x) + d_3(x) + d_4(x) + d_5(x), \quad x \in P,
\]

where \( c_i, d_i : P \rightarrow P \) are diagonalizations of some \( i \)-additive and symmetric functions \( C_i, D_i : P^i \rightarrow P \), respectively. Taking in (2) \( y = 0 \), we obtain the following formula

\[
f(x) = x[g(x) + h(x) + h(2x) + g(0)] + f(0), \quad x \in P,
\]

which used in (2) gives us

\[
x[g(x) + h(x) + h(2x) + g(0)] - y[g(y) + h(y) + h(2y) + g(0)]
\]

\[
= (x - y)[g(x) + h(x + 2y) + h(2x + y) + g(y)], \quad x, y \in P.
\]

After some simple calculations we get

\[
x[h(2x) + h(x) - h(x + 2y) - h(2x + y) - g_0(y)]
\]

\[
= y[h(2y) + h(y) - h(x + 2y) - h(2x + y) - g_0(x)], \quad x, y \in P
\]

where \( g_0(x) := g(x) - g(0), \ x \in P \).

Further, putting \( 2x \) instead of \( y \) in (10), we have

\[
h(5x) - h(4x) - h(2x) + h(x) = g_0(2x) - 2g_0(x), \quad x \neq 0,
\]

which is also satisfied for \( x = 0 \), since \( g_0(0) = 0 \). Thus

\[
h(5x) - h(4x) - h(2x) + h(x) = g_0(2x) - 2g_0(x), \quad x \in P.
\]

By (7) we obtain

\[
g_0(2x) - 2g_0(x) = 2c_2(x) + 6c_3(x) + 14c_4(x) + 30c_5(x)
\]

and similarly from (8) we have

\[
h(5x) - h(4x) - h(2x) + h(x) = 6d_2(x) + 54d_3(x) + 354d_4(x) + 2070d_5(x).
\]

Using (13) and (12) in (11) we may write

\[
6d_2(x) + 54d_3(x) + 354d_4(x) + 2070d_5(x) = 2c_2(x) + 6c_3(x) + 14c_4(x) + 30c_5(x).
\]

Comparing the corresponding terms on both sides of this equality we get

\[
c_2(x) = 3d_2(x),
\]

\[
c_3(x) = 9d_3(x),
\]

\[
7c_4(x) = 177d_4(x),
\]

\[
c_5(x) = 69d_5(x).
\]
On some equations stemming from quadrature rules

Using these equations in (7) we have
\[ g(x) = c_0 + c_1(x) + 3d_2(x) + 9d_3(x) + c_2(x) + 69d_5(x), \quad x \in P, \]  
where
\[ 7c_4(x) = 177d_4(x), \quad x \in P. \]  
(15)
Substitute in (10) \(-x\) in place of \(y\). Then
\[ h(2x) + h(-2x) - [h(x) + h(-x)] = g_0(x) + g_0(-x), \quad x \in P. \]
This, in view of (8) and (14), means that
\[ 6d_2(x) + 30d_4(x) = 6d_2(x) + 2c_4(x), \quad x \in P, \]
i.e.,
\[ c_4(x) = 15d_4(x), \quad x \in P \]
and from (15) we have
\[ d_4(x) = 0, \quad x \in P \]  
(16)
and also \(c_4 = 0\).

Now we shall show that \(d_5(x) = 0\) for all \(x \in P\). To this end we put in (10) in places of \(x\) and \(y\), respectively \(-x\) and \(2x\). Thus
\[ -2h(4x) + 3h(3x) - 2h(2x) - h(-2x) - h(-x) + 3h(0) = -g_0(2x) - 2g_0(-x) \]
for \(x \in P\). Similarly as before, using (8), (14) and (16), we have
\[ -18d_2(x) - 54d_3(x) - 1350d_5(x) = -18d_2(x) - 54d_3(x) - 2070d_5(x) \quad x \in P, \]
which means that
\[ d_5(x) = 0, \quad x \in P. \]
Now formulas (14) and (8) take forms
\[ g(x) = c_0 + c_1(x) + 3d_2(x) + 9d_3(x), \quad x \in P \]  
(17)
and
\[ h(x) = d_0 + d_1(x) + d_2(x) + d_3(x), \quad x \in P. \]  
(18)
Using these equalities in (10), we get
\[
x[-c_1(y) - 3d_1(y) + 5d_2(x) - 3d_2(y) - d_2(x + 2y) - d_2(2x + y) \\
+ 9d_3(x) - 9d_3(y) - d_3(x + 2y) - d_3(2x + y)] \\
= y[-c_1(x) - 3d_1(x) + 5d_2(y) - 3d_2(x) - d_2(x + 2y) - d_2(2x + y) \\
+ 9d_3(y) - 9d_3(x) - d_3(x + 2y) - d_3(2x + y)].
\]

Now, since the ring \(P\) is divisible by 3 and 2, the functions \(d_i\) are diagonalizations of symmetric and \(i\)-additive functions \(D_i: P^i \to P\), i.e., \(d_i(x) = D_i(x^i)\), \(x \in P\). Using these forms of \(d_i\) in the above equation we obtain
\[
2(x - y)[4D_2(x, y) + 9D_3(x, x, y) + 9D_3(x, y, y)] \\
= y[c_1(x) + 3d_1(x) + 8d_2(x) + 18d_3(x)] \\
- x[c_1(y) + 3d_1(y) + 8d_2(y) + 18d_3(y)]
\]  
(19)
for all \( x, y \in P \). Put in (19) \(-y\) instead of \( y \). Then for all \( x, y \in P \) we have

\[
2(x + y)[-4D_2(x, y) - 9D_3(x, x, y) + 9D_3(x, y, y)] \\
= -y[c_1(x) + 3d_1(x) + 8d_2(x) + 18d_3(x)] \\
- x[-c_1(y) - 3d_1(y) + 8d_2(y) - 18d_3(y)].
\]  

(20)

Adding the equations (19) and (20) we arrive at

\[
9xD_3(x, y, y) - y[4D_2(x, y) + 9D_3(x, x, y)] = -4xd_2(y), \quad x, y \in P,
\]

and, consequently,

\[
9xD_3(x, y, y) - 9yD_3(x, x, y) = 4yD_2(x, y) - 4xd_2(y), \quad x, y \in P.
\]  

(21)

Interchanging in these equations \( x \) with \( y \) and using the symmetry of both \( D_2 \) and \( D_3 \) we may write

\[
9yD_3(x, x, y) - 9xD_3(x, y, y) = 4xD_2(x, y) - 4yd_2(x), \quad x, y \in P.
\]  

(22)

Now, we add (21) and (22) to get

\[(x + y)D_2(x, y) = xd_2(y) + yd_2(x), \quad x, y \in P.\]

Put here \( x + y \) in place of \( x \), then

\[(x + 2y)D_2(x + y, y) = (x + y)d_2(y) + yd_2(x + y), \quad x, y \in P,\]

which yields

\[xD_2(x, y) = yd_2(x), \quad x, y \in P\]  

(23)

and changing the roles of \( x \) and \( y \)

\[yD_2(x, y) = xd_2(y), \quad x, y \in P.\]  

(24)

Now, we multiply (23) by \( y \) and (24) by \( x \) to obtain

\[xyD_2(x, y) = y^2d_2(x), \quad x, y \in P\]

and

\[xyD_2(x, y) = x^2d_2(y), \quad x, y \in P.\]

Thus

\[y^2d_2(x) = x^2d_2(y), \quad x, y \in P,\]

which after substituing \( y = 1 \) gives the formula

\[d_2(x) = bx^2, \quad x \in P,\]  

(25)

where \( b := d_2(1) \). Thus from (24) we obtain

\[D_2(x, y) = bxy, \quad x, y \in P.\]  

(26)
On some equations stemming from quadrature rules

Using the formulas (25) and (26) in (21) we have

$$yD_3(x, x, y) = xD_3(x, y, y), \quad x, y \in P. \quad (27)$$

Putting $x + y$ in place of $x$ (27), we get

$$yD_3(x + y, x + y, y) = (x + y)D_3(x + y, y, y),$$

which after some calculations gives

$$yD_3(x, x, y) - (x - y)D_3(x, y, y) = xd_3(y), \quad x, y \in P.$$

We use here the condition (27). Then

$$xD_3(x, y, y) - (x - y)D_3(x, y, y) = xd_3(y), \quad x, y \in P,$$

i.e.,

$$yD_3(x, y, y) = xd_3(y), \quad x, y \in P. \quad (28)$$

Clearly we also have

$$xD_3(x, x, y) = yd_3(x), \quad x, y \in P. \quad (29)$$

Now, multiply the equation (28) by $x$ and (29) by $y^2$. Then we have

$$xyD_3(x, y, y) = x^2d_3(y), \quad x, y \in P \quad (30)$$

and

$$xy^2D_3(x, x, y) = y^3d_3(x). \quad (31)$$

On the other hand, we multiply (27) by $y$. We obtain

$$y^2D_3(x, x, y) = xyD_3(x, y, y), \quad x, y \in P. \quad (32)$$

Using (32) in (30) we arrive at

$$x^2d_3(y) = y^2D_3(x, x, y), \quad x, y \in P,$$

which multiplied by $x$ yields

$$x^3d_3(y) = xy^2D_3(x, x, y), \quad x, y \in P. \quad (33)$$

Comparing the equation (31) and (33) we obtain

$$y^3d_3(x) = x^3d_3(y), \quad x, y \in P,$$

i.e.,

$$d_3(x) = ax^3, \quad x \in P, \quad (34)$$

where $a := d_3(\mathbb{1})$. Now equalities (28) and (29) take forms

$$D_3(x, y, y) = axy^2, \quad x, y \in P \quad (35)$$
and
\[ D_3(x, x, y) = ax^2y, \quad x, y \in P. \]  \hfill (36)

Using the formulas (25), (26), (34), (35) and (36) in (19) we have
\[ y[c_1(x) + 3d_1(x)] = x[c_1(y) + 3d_1(y)], \quad x, y \in P. \]

Substituting here \( y = 1 \) we obtain
\[ c_1(x) + 3d_1(x) = x[c_1(1) + 3d_1(1)], \quad x \in P, \]

which means that
\[ c_1(x) = cx - 3d_1(x), \quad x \in P, \]

where \( c := c_1(1) + 3d_1(1) \).

Thus we have shown that the formulas (17) and (18) may be written in the form
\[ g(x) = 9ax^3 + 3bx^2 + cx - 3d_1(x) + c_0, \quad x \in P \]

and
\[ h(x) = ax^3 + bx^2 + d_1(x) + d_0, \quad x \in P, \]

where \( d_1 \) is a given additive function. Now it suffices to use the obtained expressions in (9), to get the desired formula for \( f \).

It is an easy calculation to show that these functions \( f, g, h \) satisfy the equation (2).

With the aid of this theorem we may prove also a Stamate-kind result.

Corollary 1
Let \( P \) be an integral domain with unit element \( 1 \), uniquely divisible by \( 5! \) and such that for every \( n \in \mathbb{N} \) we have \( n1 \neq 0 \). Functions \( f, g, h: P \to P \) satisfy the equation (3) if and only if there exist \( a, \bar{a}, b, c, d, \bar{d} \in P \) and an additive function \( A: P \to P \) such that
\[
\begin{align*}
    f(x) &= \begin{cases} 
        18ax^3 + 8bx^2 + cx + 2d, & x \neq 0 \\
        \bar{a}, & x = 0 
    \end{cases}, \\
    g(x) &= \begin{cases} 
        -9ax^3 - 5bx^2 - 3A(x) - d - \bar{d}, & x \neq 0 \\
        d - \bar{d} - \bar{a}, & x = 0 
    \end{cases}, \\
    h(x) &= ax^3 + bx^2 + A(x) + \bar{d}, \quad x \in P.
\end{align*}
\]

Conversely, \( f, g, h: P \to P \) given by the above equalities satisfy (2).

Proof. First we write the equation (3) in the form
\[
(y - x)f(y) - yf(y) + (y - x)f(x) + xf(x) = (x - y)[g(x) + h(2x + y) + h(x + 2y) + g(y)]
\]
On some equations stemming from quadrature rules

and, consequently,

\[ xf(x) - yf(y) = (x - y)[g(x) + f(x) + h(2x + y) + h(x + 2y) + g(y) + f(y)]. \]

Putting here \( k(t) := g(t) + f(t) \) and \( F(t) := tf(t) \) for all \( t \in P \) we obtain

\[ F(x) - F(y) = (x - y)[k(x) + h(2x + y) + h(x + 2y) + k(y)], \quad x, y \in P. \]

Thus, using Theorem 1, we get

\[ \begin{align*}
    xf(x) &= 18ax^4 + 8bx^3 + cx^2 + 2dx + e, & x \in P, \\
    g(x) + f(x) &= 9ax^3 + 3bx^2 + cx - 3A(x) + d - \bar{d}, & x \in P, \\
    h(x) &= ax^3 + bx^2 + A(x) + \bar{d}, & x \in P.
\end{align*} \tag{37} \tag{38} \tag{39} \]

Now, from (37) it easily follows that \( e = 0 \) and furthermore

\[ xf(x) = 18ax^4 + 8bx^3 + cx^2 + 2dx, \]

i.e.,

\[ f(x) = 18ax^3 + 8bx^2 + cx + 2d, \quad x \neq 0, \]

which gives us

\[ g(x) = -9ax^3 - 5bx^2 - 3A(x) - d - \bar{d}, \quad x \neq 0. \]

Moreover, from (38) we get \( g(0) + f(0) = d - \bar{d} \), thus putting \( \bar{a} := f(0) \) we obtain that \( g(0) = d - \bar{d} - \bar{a} \).

On the other hand, it is easy to see that functions given by the above formulae yield a solution of the equation (3).

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**References**


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