Dedicated to Professor Andrzej Zajtz
on the occasion of his 70th birthday

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Local analytic solutions of a functional equation

Abstract. All analytic solutions of the functional equation

\[ |f(r \exp(i\theta))|^2 + |f(1)|^2 = |f(r)|^2 + |f(\exp(i\theta))|^2 \]

in the annulus

\[ P := \{ z \in \mathbb{C} : 1 - \epsilon < |z| < 1 + \epsilon \} \]

and in the domain

\[ D := \{ z = re^{i\theta} \in \mathbb{C} : 1 - \epsilon < r < 1 + \epsilon, \ \theta \in (-\delta, \delta) \}, \]

defined in the annulus

\[ P := \{ z \in \mathbb{C} : 1 - \epsilon < |z| < 1 + \epsilon \} \]

are found.

1. Introduction

Hiroshi Haruki in [1] studied the following functional equations

\[ |f(r \exp(i\theta))|^2 + |f(1)|^2 = |f(r)|^2 + |f(\exp(i\theta))|^2, \] (1)

and

\[ |f(r \exp(i\theta))| = |f(r)|, \] (2)

where \( r > 0, \ \theta \) are real. Equation (1) can be obtained from (2). In fact, let us put \( r = 1 \) in (2). Then we have

\[ |f(\exp(i\theta))| = |f(1)| \] (3)

for \( \theta \in \mathbb{R} \). Next squaring (2) and (3) and adding them together we infer (1). Thus (1) is a generalization of (2), i.e., if \( f \) is a solution of (2), then it is a solution of (1). In paper [1] H. Haruki showed that all analytic solutions in \( \mathbb{C} \setminus \{0\} \) of (1) which are analytic at 0 or have a pole at this point can be written as follows

\[ f(z) = Az^p + Bz^{-p}, \] (4)

where \( A, \ B \) are complex constants and \( p \) is an integer.

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We are going to prove that the functions of the form (4) are unique analytic solutions of (1) in the annulus
\[ P := \{ z \in \mathbb{C} : 1 - \epsilon < |z| < 1 + \epsilon \}, \]
where \( 0 < \epsilon \leq 1 \) is a constant. We shall also find all analytic solutions of (1) in the domain
\[ D := \{ z = re^{i\theta} \in \mathbb{C} : 1 - \epsilon < r < 1 + \epsilon, \ \theta \in (-\delta, \delta) \}, \]
where \( 0 < \epsilon \leq 1 \) and \( 0 < \delta \leq \pi \) are given constants. Moreover, we shall determine all analytic solutions in \( P \) and in \( D \) of (2) and of the equation
\[ |f(r \exp(i\theta))| = |f(\exp(i\theta))|. \tag{5} \]
Of course, (1) is also a generalization of (5).

2. Solutions of (1), (2) and (5) in \( P \)

In this section we will be concerned with analytic solutions of equations (1), (2) and (5) in the annulus \( P \).

**Theorem 1**

If \( f \) is an analytic solution of (1) in \( P \), then there exist complex constants \( A, B \) and an integer \( p \) such that (4) is valid. Conversely, for every complex constants \( A, B \) and for every integer \( p \), \( f \) given by (4) is a solution of (1).

**Proof.** It is easy to check that \( f \) given by (4) satisfies (1). The function \( f(z) \equiv 0 \) in \( P \) is a solution of (1) of the form (4). Suppose that an analytic function \( f \) is a solution of (1) and \( f \not\equiv 0 \). Of course,
\[ f(re^{i\theta} f(re^{i\theta}) + |f(1)|^2 = |f(r)|^2 + |f(e^{i\theta})|^2 \tag{6} \]
for \( \theta \in \mathbb{R} \) and \( r \in (1 - \epsilon, 1 + \epsilon) \). Differentiating (6) at first with respect to \( r \) and then with respect to \( \theta \) we successively infer
\[ e^{i\theta} f'(re^{i\theta}) f(re^{i\theta}) + e^{-i\theta} f(re^{i\theta}) f'(re^{i\theta}) = \frac{d}{dr} |f(r)|^2 \]
and
\[ re^{2i\theta} f''(re^{i\theta}) f(re^{i\theta}) - re^{-2i\theta} f(re^{i\theta}) f''(re^{i\theta}) + e^{i\theta} f'(re^{i\theta}) f'(re^{i\theta}) \]
\[ - e^{-i\theta} f'(re^{i\theta}) f'(re^{i\theta}) \]
\[ = 0. \]
Let us multiply the obtained equality by \( r \) and replace \( re^{i\theta} \) by \( z \). Then
\[ z^2 f''(z)f(z) - \overline{z^2 f''(z)f(z)} + z f'(z)f(z) - \overline{z f'(z)f(z)} = 0, \]

i.e.,
\[ \Im \left[ z^2 f''(z)f(z) + z f'(z)f(z) \right] = 0 \]  
(7)
for all \( z \in P \). Since \( f \neq 0 \), we can find a disc \( V \subset P \) such that \( f(z) \neq 0 \) for all \( z \in V \). The equality \( f(z) = \frac{|f(z)|^2}{f(z)} \), valid in this disc, and (7) imply
\[ \Im \left[ \frac{z^2 f''(z) + zf'(z)}{f(z)} \right] = 0 \]
for all \( z \in V \). Since an analytic function preserves domains, there exists a real constant \( k \) such that
\[ z^2 f''(z) + zf'(z) - kf(z) = 0 \]  
(8)
for all \( z \in V \). By the Identity Theorem formula (8) remains valid in \( P \). (The above part of the proof is due to H. Haruki, see [1], pp. 130-131). We can find complex numbers \( a_n, n \in \mathbb{Z} \) such that for all \( z \in P \),
\[ f(z) = \sum_{n=-\infty}^{\infty} a_n z^n. \]
Since
\[ f'(z) = \sum_{n=-\infty}^{\infty} n a_n z^{n-1}, \quad f''(z) = \sum_{n=-\infty}^{\infty} n(n-1) a_n z^{n-2} \]
we conclude that
\[ 0 = z^2 f''(z) + zf'(z) - kf(z) = \sum_{n=-\infty}^{\infty} [n(n-1) + n - k]a_n z^n, \]
whence
\[ (n^2 - k)a_n = 0 \quad \text{for all } n \in \mathbb{Z}. \]  
(9)
We choose \( p \in \mathbb{Z} \) such that \( a_p \neq 0 \). It is possible as \( f \neq 0 \). From (9) we get that \( p^2 = k \) and
\[ (n^2 - p^2) a_n = 0 \quad \text{for all } n \in \mathbb{Z}. \]
So, if \( n^2 \neq p^2 \), then \( a_n = 0 \), whence it follows that \( a_n = 0 \) for all \( n \neq p \) and \( n \neq -p \). Thus
\[ f(z) = a_p z^p + a_{-p} z^{-p} \]
for \( z \in P \), as desired.

The following two lemmas are quite obvious.

**Lemma 1**
If the equality
\[ Ae^{ia\theta} + \overline{A}e^{-ia\theta} = A + \overline{A} \]
for all \( \theta \).
holds true for all \( \theta \in (\delta, \bar{\delta}) \), where \( A \) is a complex constant, \( a \neq 0 \) is a real one, then \( A = 0 \).

**Lemma 2**

If the equality

\[
\alpha e^{a\theta} + \beta e^{-a\theta} = \alpha + \beta
\]

holds true for all \( \theta \in (\delta, \bar{\delta}) \), where \( a \neq 0 \), \( \alpha, \beta \) are real constants, then \( \alpha = \beta = 0 \).

Now we will consider equation (2). As we mentioned above, every solution of (2) is a solution of (1). Thus if \( f \) is an analytic solution of (2), then \( f \) has to be of form (4) for some complex constants \( A, B \) and some integer \( p \). Assume that \( p \neq 0 \). Substituting (4) to (2) we get

\[
ABe^{2ip\theta} + \overline{A}Be^{-2ip\theta} = AB + \overline{A}B, \quad \theta \in \mathbb{R}.
\]

Lemma 1 yields \( A = 0 \) or \( B = 0 \). Thus we have

**Theorem 2**

If \( f \) is an analytic solution of (2) in the annulus \( P \), then there exist a complex constant \( A \) and an integer \( p \) such that

\[
f(z) = Az^p.
\]

Conversely, for every complex constant \( A \) and for every integer \( p \), the function \( f \) given by (10) is a solution of (2).

**Theorem 3**

Every analytic solution of (5) in the annulus \( P \) is a constant function.

*Proof.* Suppose that \( f \) is a solution of (5). Then \( f \) has to be of form (4). We may assume that \( p \neq 0 \). Combining (4) with (5) we obtain

\[
|A|^2r^{2p} + |B|^2r^{-2p} = |A|^2 + |B|^2 \quad \text{for all } r \in (1 - \epsilon, 1 + \epsilon).
\]

Lemma 2 shows that \( A = B = 0 \), which completes the proof.

### 3. Solutions of (1), (2) and (5) in \( D \)

In this part of the paper we shall find all analytic solutions of equations (1), (2) and (5) in the domain \( D := \{re^{i\theta} : 1 - \epsilon < r < 1 + \epsilon, \theta \in (-\delta, \delta)\} \), where \( 0 < \epsilon \leq 1 \) and \( 0 < \delta \leq \pi \). In the sequel \( z^\alpha \) denotes the principal branch
of the power in $D$ and $\log z$ is the principal branch of the logarithm of $z$, i.e., $z^a = \exp(a \log z)$ and $\log z = \log |z| + i \arg z$ for $z \in D$, where $\arg z \in (-\delta, \delta)$.

**Theorem 4**

If an analytic function $f$ satisfies (1) in $D$, then there exist complex constants $A, B$ and $a \in \mathbb{R}$ or $a \in i\mathbb{R}$ such that

$$f(z) = Az^a + Bz^{-a}.$$  

(11)

Conversely, every function $f$ of form (11) with arbitrary complex constants $A, B$ and arbitrary real or purely imaginary constant $a$ is a solution of (1).

**Proof.** We may repeat the argument of the proof of Theorem 1. Thus we observe that if an analytic function $f$ satisfies (1) in $D$, then it has to be a solution of the differential equation

$$z^2 f''(z) + zf'(z) - kf(z) = 0, \quad z \in D,$$

(12)

where $k$ is a real constant. Let

$$G = \{ \log z : z \in D \}.$$  

Of course, $G$ is a domain. We define a function $g : G \rightarrow \mathbb{C}$ as follows

$$g(u) := f(e^u).$$

$g$ is analytic, $f(z) = g(\log z)$ for $z \in D$ and

$$e^u f'(e^u) = g'(u), \quad e^{2u} f''(e^u) = g''(u) - g'(u), \quad u \in G.$$  

(13)

It follows from (12) that

$$e^{2u} f''(e^u) + e^u f'(e^u) - kf(e^u) = 0 \quad \text{for all } u \in G,$$

whence by (13)

$$g''(u) - kg(u) = 0, \quad u \in G.$$  

Solving this differential equation we get

$$g(u) = Ae^{au} + Be^{-au},$$

where $A, B$ are suitable complex constants and $a^2 = k$. So $a$ is a real constant or $a = ic$, where $c \in \mathbb{R}$. Putting $u = \log z$ we obtain (11). The first assertion of the theorem follows.

For the second conclusion, let us take arbitrarily $a \in \mathbb{R}, A, B \in \mathbb{C}$ and let $f$ be given by (11). We observe that

$$f(re^{i\theta}) = Ar^a e^{ia} + Br^{-a} e^{-ia}; \quad f(e^{i\theta}) = Ae^{ia} + Be^{-ia},$$  

$$f(r) = Ar^a + Br^{-a}; \quad f(1) = A + B.$$  

Thus
All analytic solutions of Theorem 5

Since (1) is a generalization of (2) we can apply Theorem 4. Thus there exist complex constants $A, B$ and real or purely imaginary $a \neq 0$ such that $f$ is given by (11). At first we assume that $a$ is real. Substituting (11) in (2) after some easier calculations we obtain

$$|f(re^{i\theta})|^2 + |f(1)|^2$$

$$= (Ar^a e^{i\theta a} + Br^{-a} e^{-i\theta a})(\overline{Ar}^a e^{-i\theta a} + \overline{Br}^{-a} e^{i\theta a}) + (A + B)(\overline{A} + \overline{B})$$

$$= |A|^2 r^{2a} + |B|^2 r^{-2a} + \overline{AB} e^{2i\theta a} + \overline{AB} e^{-2i\theta a} + |A|^2 + |B|^2 + \overline{AB} + \overline{AB}$$

and

$$|f(e^{i\theta})|^2 + |f(r)|^2$$

$$= (Ar^a e^{i\theta a} + Br^{-a} e^{-i\theta a})(\overline{Ar}^a e^{-i\theta a} + \overline{Br}^{-a} e^{i\theta a}) + (Ar^a + Br^{-a})(\overline{A}r^a + \overline{B}r^{-a})$$

$$= |A|^2 + |B|^2 + \overline{AB} e^{2i\theta a} + \overline{AB} e^{-2i\theta a} + |A|^2 r^{2a} + |B|^2 r^{-2a} + \overline{AB} + \overline{AB}.$$ 

Now we assume that $a = ic$, where $c \in \mathbb{R}$. Then

$$f(re^{i\theta}) = Ae^{ic\log r + i\theta} + Be^{-ic\log r + i\theta}$$

$$= Ae^{-c} e^{ic\log r} + Be^{c} e^{-ic\log r},$$

$$f(e^{i\theta}) = Ae^{-c} + Be^{c},$$

$$f(r) = Ae^{ic\log r} + Be^{-ic\log r},$$

$$f(1) = A + B.$$ 

These formulas lead to

$$|f(re^{i\theta})|^2 + |f(1)|^2$$

$$= (Ae^{-c} e^{ic\log r} + Be^{c} e^{-ic\log r})(\overline{A}e^{-c} e^{-ic\log r} + \overline{B}e^{c} e^{ic\log r}) + |A + B|^2$$

$$= |A|^2 e^{-2c\theta} + |B|^2 e^{2c\theta} + \overline{AB} e^{2ic\log r} + ABe^{-2ic\log r}$$

$$+ |A|^2 + |B|^2 + \overline{AB} + \overline{AB}$$

and

$$|f(e^{i\theta})|^2 + |f(r)|^2$$

$$= (Ae^{-c} + Be^{c})(\overline{A}e^{-c} + \overline{B}e^{c})$$

$$+ (Ae^{ic\log r} + Be^{-ic\log r})(\overline{A}e^{-ic\log r} + \overline{B}e^{ic\log r})$$

$$= |A|^2 e^{-2c\theta} + |B|^2 e^{2c\theta} + \overline{AB} + \overline{AB} + |A|^2 + |B|^2$$

$$+ ABe^{2ic\log r} + ABe^{-2ic\log r}.$$ 

So in both cases the function $f$ given by (11) satisfies (1), as required.

**Theorem 5**

All analytic solutions of (2) in $D$ are of the form

$$f(z) = Az^a,$$ 

where $A$ is a complex constant and $a$ is a real one.

**Proof.** Suppose that $f$ is a non-constant analytic solution of (2) in $D$. Since (1) is a generalization of (2) we can apply Theorem 4. Thus there exist complex constants $A, B$ and real or purely imaginary $a \neq 0$ such that $f$ is given by (11). At first we assume that $a$ is real. Substituting (11) in (2) after some easy calculations we obtain
\[
\overline{AB} \exp(-2ia\theta) + A\overline{B} \exp(2ia\theta) = \overline{AB} + A\overline{B}
\]
for \( \theta \in (-\delta, \delta) \). Lemma 1 yields \( A = 0 \) or \( B = 0 \) and \( f \) is of the form (14), as required.

Now, we assume that \( a = ic \), where \( c \) is real. Replacing in (2), \( f(z) \) by (11) we infer the equality

\[
|A|^2 \exp(-2c\theta) + |B|^2 \exp(2c\theta) = |A|^2 + |B|^2.
\]
This together with Lemma 2 yields \( A = B = 0 \).

**Theorem 6**

All analytic solutions of (5) in \( D \) are given by the formula

\[
f(z) = Az^{ic},
\]
where \( A \) is a complex constant and \( c \) is a real one.

**Proof.** We argue as in the preceding proof. Suppose that \( f \) is a non-constant analytic solution of (5) in \( D \). \( f \) has to be given by (11). Assume that \( a \) is a real constant. Substituting (11) in (5) we get

\[
|A|^2 r^{2a} + |B|^2 r^{-2a} = |A|^2 + |B|^2
\]
for all \( r \in (1-\epsilon, 1+\epsilon) \). From Lemma 2 we infer that \( A = B = 0 \). It remains to consider \( a = ic \), where \( c \) is real. Again substituting (11) in (5) we can obtain

\[
A\overline{B} \exp(2ic \log r) + \overline{A}B \exp(-2ic \log r) = A\overline{B} + \overline{A}B.
\]
The above formula and Lemma 1 yield (15).

**References**


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