Abstract. In this paper we define strong ideals and horizontal ideals in pseudo-BCH-algebras and investigate the properties and characterizations of them.

1. Introduction

In 1966, Y. Imai and K. Iséki (13, 14) introduced BCK- and BCI-algebras. In 1983, Q.P. Hu and X. Li (11) introduced BCH-algebras. It is known that BCK- and BCI-algebras are contained in the class of BCH-algebras.

In 2001, G. Georgescu and A. Iorgulescu (10) introduced pseudo-BCK-algebras as an extension of BCK-algebras. In 2008, W.A. Dudek and Y.B. Jun (2) introduced pseudo-BCI-algebras as a natural generalization of BCI-algebras and of pseudo-BCK-algebras. These algebras have also connections with other algebras of logic such as pseudo-MV-algebras and pseudo-BL-algebras defined by G. Georgescu and A. Iorgulescu in [8] and [9], respectively. Those algebras were investigated by several authors in a number of papers (see for example [3, 5, 6, 7, 15, 17, 18, 19]). Recently, A. Walendziak (20) introduced pseudo-BCH-algebras as an extension of BCH-algebras and studied the set Cen X of all minimal elements of a pseudo-BCH-algebra X, the so-called centre of X. He also considered ideals in pseudo-BCH-algebras and established a relationship between the ideals of a pseudo-BCH-algebra and the ideals of its centre.

In this paper we define strong ideals and horizontal ideals in pseudo-BCH-algebras and investigate the properties and characterizations of them.

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2. Preliminaries

We recall that an algebra $\mathfrak{X} = (X; *, 0)$ of type $(2, 0)$ is called a $BCH$-algebra if it satisfies the following axioms:

(BCH-1) $x * x = 0$;
(BCH-2) $(x * y) * z = (x * z) * y$;
(BCH-3) $x * y = y * x = 0 \implies x = y$.

A BCH-algebra $\mathfrak{X}$ is said to be a $BCI$-algebra if it satisfies the identity

(BCI) $((x * y) * (x * z)) * (z * y) = 0$.

A $BCK$-algebra is a $BCI$-algebra $\mathfrak{X}$ satisfying the law $0 * x = 0$.

Definition 2.1 ([2])
A pseudo-BCI-algebra is a structure $\mathfrak{X} = (X; \le, *, \cdot, 0)$, where $\le$ is a binary relation on the set $X$, $*$ and $\cdot$ are binary operations on $X$ and $0$ is an element of $X$, satisfying the axioms:

(pBCI-1) $(x * y) \cdot (x * z) \le z \cdot y$, $(x \cdot y) * (x \cdot z) \le z \cdot y$;
(pBCI-2) $x * (x \cdot y) \le y$, $x \cdot (x * y) \le y$;
(pBCI-3) $x \le x$;
(pBCI-4) $x \le y, y \le x \implies x = y$;
(pBCI-5) $x \le y \iff x * y = 0 \iff x \cdot y = 0$.

A pseudo-BCI-algebra $\mathfrak{X}$ is called a pseudo-BCK-algebra if it satisfies the identities

(pBCK) $0 * x = 0 \cdot x = 0$.

Definition 2.2 ([20])
A pseudo-BCH-algebra is an algebra $\mathfrak{X} = (X; *, \cdot, 0)$ of type $(2, 2, 0)$ satisfying the axioms:

(pBCH-1) $x * x = x \cdot x = 0$;
(pBCH-2) $(x * y) \cdot z = (x \cdot z) * y$;
(pBCH-3) $x * y = y \cdot x = 0 \implies x = y$;
(pBCH-4) $x * y = 0 \iff x \cdot y = 0$.

We define a binary relation $\le$ on $X$ by

$x \le y \iff x * y = 0 \iff x \cdot y = 0$.

Throughout this paper $\mathfrak{X}$ will denote a pseudo-BCH-algebra.

Remark
Observe that if $(X; *, 0)$ is a BCH-algebra, then letting $x \cdot y := x * y$, produces a pseudo-BCH-algebra $(X; *, \cdot, 0)$. Therefore, every BCH-algebra is a pseudo-BCH-algebra in a natural way. It is easy to see that if $(X; *, 0)$ is a pseudo-BCH-algebra, then $(X; \cdot, *, 0)$ is also a pseudo-BCH-algebra. From Proposition 3.2 of [2] we conclude that if $(X; \le, *, 0)$ is a pseudo-BCI-algebra, then $(X; *, 0)$ is a pseudo-BCH-algebra.
Example 2.3 ([21])
Let \((G; \cdot, e)\) be a group. Define binary operations \(*\) and \(\diamond\) on \(G\) by
\[
a * b = ab^{-1} \quad \text{and} \quad a \diamond b = b^{-1}a
\]
for all \(a, b \in G\). Then \(\mathcal{G} = (G; *, \diamond, e)\) is a pseudo-BCH-algebra.

We say that a pseudo-BCH-algebra \(X\) is proper if \(* \neq \diamond\) and \(X\) is not a pseudo-
BCI-algebra.

Example 2.4
Consider the set \(X = \{0, a, b, c, d, e, f, g, h\}\) with the operations \(*\) and \(\diamond\) defined
by the following tables:

\[
\begin{array}{cccccccc}
* & 0 & a & b & c & d & e & f & g \\
0 & 0 & 0 & 0 & 0 & d & e & f & h \\
a & a & 0 & c & c & d & e & f & h \\
b & b & 0 & 0 & b & d & e & f & h \\
c & c & 0 & 0 & 0 & d & e & f & h \\
d & d & d & d & d & 0 & h & g & e \\
e & e & e & e & e & g & 0 & h & f \\
f & f & f & f & f & h & g & 0 & d \\
g & g & g & g & g & e & f & d & 0 \\
h & h & h & h & h & f & d & e & g
\end{array}
\]

and

\[
\begin{array}{cccccccc}
\diamond & 0 & a & b & c & d & e & f & g \\
0 & 0 & 0 & 0 & 0 & d & e & f & h \\
a & a & 0 & c & c & d & e & f & h \\
b & b & 0 & 0 & b & d & e & f & h \\
c & c & 0 & 0 & 0 & d & e & f & h \\
d & d & d & d & d & 0 & h & g & f \\
e & e & e & e & e & g & 0 & h & d \\
f & f & f & f & f & h & g & 0 & e \\
g & g & g & g & g & e & f & d & 0 \\
h & h & h & h & h & f & d & e & h
\end{array}
\]

Then \((X; *, \diamond, 0)\) is a proper pseudo-BCH-algebra (see [21]).

From [20] it follows that in any pseudo-BCH-algebra \(X\) for all \(x, y \in X\) we have:

(a1) \(x * (x \diamond y) \leq y\) and \(x \diamond (x * y) \leq y\);

(a2) \(x * 0 = x \diamond 0 = x\);

(a3) \(0 * x = 0 \diamond x\);

(a4) \(0 * (0 * (0 * x)) = 0 * x\);

(a5) \(0 * (x * y) = (0 * x) \diamond (0 * y)\);

(a6) \(0 * (x \diamond y) = (0 * x) * (0 * y)\).

Following the terminology of [20], the set \(\{a \in X : a = 0 * (0 * a)\}\) will be called the
centre of \(X\). W shall denote it by \(\text{Cen } X\). By Proposition 4.1 of [20], \(\text{Cen } X\) is
the set of all minimal elements of \( X \), that is, \( \text{Cen} \, X = \{ a \in X : \forall x \in X \ x \leq a \implies x = a \} \).

**Example 2.5**
Let \( X = (X; \ast, \circ, 0) \) be the pseudo-BCH-algebra given in Example 2.4. It is easily seen that \( \text{Cen} \, X = \{ 0, d, e, f, g, h \} \).

**Proposition 2.6** ([20])
Let \( X \) be a pseudo-BCH-algebra, and let \( a \in X \). Then the following conditions are equivalent:

(i) \( a \in \text{Cen} \, X \).
(ii) \( a \ast x = 0 \ast (x \ast a) \) for all \( x \in X \).
(iii) \( a \circ x = 0 \ast (x \circ a) \) for all \( x \in X \).

**Proposition 2.7** ([20])
\( \text{Cen} \, X \) is a subalgebra of \( X \).

**Definition 2.8**
A subset \( I \) of \( X \) is called an *ideal* of \( X \) if it satisfies for all \( x, y \in X \):

(I1) \( 0 \in I \);
(I2) if \( x \ast y \in I \) and \( y \in I \), then \( x \in I \).

We will denote by \( \text{Id}(X) \) the set of all ideals of \( X \). Obviously, \( \{0\}, X \in \text{Id}(X) \).

**Proposition 2.9** ([20])
Let \( I \) be an ideal of \( X \). For any \( x, y \in X \), if \( y \in I \) and \( x \leq y \), then \( x \in I \).

**Proposition 2.10** ([20])
Let \( X \) be a pseudo-BCH-algebra and \( I \) be a subset of \( X \) satisfying (I1). Then \( I \) is an ideal of \( X \) if and only if for all \( x, y \in X \),

(I2') if \( x \circ y \in I \) and \( y \in I \), then \( x \in I \).

**Proposition 2.11**
Let \( I \) be an ideal of \( X \) and \( x \in I \). Then \( 0 \ast (0 \ast x) \in I \).

**Proof.** Let \( x \in X \). From (a3) and (a1) it follows that \( 0 \ast (0 \ast x) = 0 \ast (0 \circ x) \leq x \). Since \( x \in I \), by Proposition 2.9 \( 0 \ast (0 \ast x) \in I \).

**Example 2.12**
Consider the pseudo-BCH-algebra \( \mathfrak{G} \), which is given in Example 2.3. Let \( a \) be an element of \( G \). It is routine to verify that \( \{ a^n : n \in \mathbb{N} \cup \{0\} \} \) is an ideal of \( \mathfrak{G} \).

**Proposition 2.13** ([21])
Let \( X \) be a pseudo-BCH-algebra and \( I \) be a subset of \( X \) containing \( 0 \). The following statements are equivalent:

(i) \( I \) is an ideal of \( X \).
(ii) \( x \in I, \ y \in X - I \implies y \ast x \in X - I \).
(iii) \( x \in I, \ y \in X - I \implies y \circ x \in X - I \).
For any pseudo-BCH-algebra $X$, we set $K(X) = \{x \in X : 0 \leq x\}$.

From \cite{20} it follows that $K(X)$ is a subalgebra of $X$. Observe that

$$\text{Cen } X \cap K(X) = \{0\}. \quad (1)$$

Indeed, $0 \in \text{Cen } X \cap K(X)$ and if $x \in \text{Cen } X \cap K(X)$, then $x = 0 \ast (0 \ast x) = 0 \ast 0 = 0$.

### 3. Closed, strong, and horizontal ideals

An ideal $I$ of $X$ is said to be **closed** if $0 \ast x \in I$ for every $x \in I$.

**Proposition 3.1** \cite{20}

An ideal $I$ of $X$ is closed if and only if $I$ is a subalgebra of $X$.

**Proposition 3.2** \cite{20}

Every ideal of a finite pseudo-BCH-algebra is closed.

**Example 3.3**

Let $M$ be the set of all matrices of the form $A = \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}$, where $x$ and $y$ are rational numbers such that $x > 0$. Evidently, $(M; \cdot, E)$, where $\cdot$ is the usual multiplication of matrices and $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, is a group. We define the binary operations $\ast$ and $\cdot$ on $M$ by

$$A \ast B = AB^{-1} \quad \text{and} \quad A \cdot B = B^{-1}A$$

for all $A, B \in M$. Then $M = (M; \ast, \cdot, E)$ is a pseudo-BCH-algebra (by Example 2.3). Let $C = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. The set $I = \{C^n : n \in \mathbb{N} \cup \{0\}\}$ is an ideal of $M$ (see Example 2.12). Observe that $I$ is not closed. Indeed, $E \ast C = EC^{-1} = C^{-1} \notin I$.

**Proposition 3.4** \cite{20}

$K(X)$ is a closed ideal.

**Definition 3.5**

An ideal $I$ of a pseudo-BCH-algebra $X$ is called **strong** if, for all $x, y \in X$, $x \in I$ and $y \in X - I$ imply $x \ast y \in X - I$.

It is clear that $X$ is a strong ideal of $X$. Note that in BCI-algebras such ideals were investigated in \cite{11} (see also \cite{12}).

**Theorem 3.6**

Let $I$ be an ideal of $X$. Then the following statements are equivalent:

(i) $I$ is strong.

(ii) For any $x, y \in X$, $x \ast y \in I$ and $x \in I$ imply $y \in I$. 
(iii) For every $x \in X$ both $x$ and $0 \ast x$ belong or not belong to $I$.
(iv) For every $x \in X$, $0 \ast x \in I$ implies $x \in I$.

Proof. (i) Let $I$ be a strong ideal. Let $x \in I$ and $x \ast y \in I$. Suppose that $y \notin I$. By the strongness of $I$, $x \ast y \in X - I$. This is a contradiction.

(ii) Let $x \in I$. Then, by Proposition 2.11, $0 \ast (0 \ast x) \in I$. Since $0 \notin I$, according to (ii) we deduce $0 \ast x \in I$. Thus, if $x \in I$, then $0 \ast x \in I$. Suppose now that $x \notin I$ and $0 \ast x \in I$. Applying (pBCH-2) and (pBCH-1) we have

$$[(0 \ast x) \ast x] \circ (0 \ast x) = ((0 \ast x) \circ (0 \ast x)) \ast x = 0 \ast x \in I,$$

and from the definition of ideal we conclude that $(0 \ast x) \ast x \in I$. By (ii) $x \in I$, which is a contradiction. Thus, if $x \notin I$, then $0 \ast x \notin I$.

(iii) Obvious.

(iv) Any ideal $I$ with the property that both $x$ and $0 \ast x$ belong or not belong to $I$, is obviously closed. To prove that $I$ is strong, let $x \in I$ and $y \in X - I$. On the contrary, assume that $x \ast y \in I$. Hence $0 \ast (x \ast y) \in I$, and by (a3) we obtain $(0 \ast x) \circ (0 \ast y) \in I$. Also $0 \ast x \in I$. Since $I$ is a subalgebra of $\mathcal{X}$ (by Proposition 3.1) it follows that $((0 \ast x) \circ (0 \ast y)) \ast (0 \ast x) \in I$. Then $0 \ast (0 \ast y) \in I$, because

$$0 \ast (0 \ast y) = 0 \circ (0 \ast y)$$
$$= ((0 \ast x) \ast (0 \ast x)) \circ (0 \ast y) \quad \text{by (a3)}$$
$$= ((0 \ast x) \circ (0 \ast y)) \ast (0 \ast x). \quad \text{by (pBCH-1)}$$

Using (iv) we conclude that $y \in I$, a contradiction.

From the proof of Theorem 3.6 we have the following corollaries.

**Corollary 3.7**
Every strong ideal of $\mathcal{X}$ is closed.

**Corollary 3.8**
Let $I$ be an ideal of $\mathcal{X}$. Then the following statements are equivalent:

(i) $I$ is strong.

(ii) For any $x, y \in X$, $x \in I$ and $y \in X - I$ imply $x \circ y \in X - I$.

(iii) For any $x, y \in X$, $x \circ y \in I$ and $x \in I$ imply $y \in I$.

Combining Proposition 2.13 and Corollary 3.8 we get

**Theorem 3.9**
Let $\mathcal{X}$ be a pseudo-BCH-algebra and $I$ be a subset of $X$ containing $0$. The following statements are equivalent:

(i) $I$ is a strong ideal of $\mathcal{X}$.

(ii) For any $x, y \in X$, $x \in I$ and $y \in X - I$ imply $x \ast y, y \ast x \in X - I$.

(iii) For any $x, y \in X$, $x \in I$ and $y \in X - I$ imply $x \circ y, y \circ x \in X - I$. 
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Theorem 3.10
Let \( I \) be a closed ideal of \( X \). Then the following statements are equivalent:

(i) \( I \) is strong.
(ii) For all \( x, y \in X \), \( x \leq y \) and \( x \in I \) imply \( y \in I \).
(iii) For every \( x \in X \), \( 0 \ast (0 \ast x) \in I \) implies \( x \in I \).

Proof. (i) \iff (ii) Let \( I \) be a strong ideal. Let \( x \leq y \) and \( x \in I \). Then \( x \ast y = 0 \in I \).

As \( I \) satisfies (ii) of Theorem 3.6 we get \( y \in I \).

(ii) \iff (iii) Let \( 0 \ast (0 \ast x) \in I \). Since \( 0 \ast (0 \ast x) \leq x \), applying (ii) we see that \( x \in I \).

(iii) \iff (i) Let \( 0 \ast x \in I \). Then \( 0 \ast (0 \ast x) \in I \), because \( I \) is closed. From (iii) it follows that \( x \in I \). Thus condition (iv) of Theorem 3.6 holds. Consequently, \( I \) is a strong ideal.

As a consequence of Proposition 3.2 and Theorem 3.10 we get the following

Proposition 3.11
Let \( I \) be an ideal of a finite pseudo-BCH-algebra \( X \). Then the following statements are equivalent:

(i) \( I \) is strong.
(ii) For all \( x, y \in X \), \( x \leq y \) and \( x \in I \) imply \( y \in I \).
(iii) For every \( x \in X \), \( 0 \ast (0 \ast x) \in I \) implies \( x \in I \).

Proposition 3.12
\( K(X) \) is a strong ideal.

Proof. By Proposition 3.4, \( K(X) \) is closed. Let \( 0 \ast (0 \ast x) \in K(X) \). Then \( 0 \ast (0 \ast (0 \ast x)) = 0 \). Since \( 0 \ast (0 \ast (0 \ast x)) = 0 \ast x \) (see [a4]), we have \( 0 \ast x = 0 \). Hence \( x \in K(X) \), and thus \( K(X) \) satisfies condition (iv) of Theorem 3.6 Therefore \( K(X) \) is strong.

Proposition 3.13
Let \( I \in \text{Id}(X) \). If \( I \subset K(X) \), then \( I \) is not a strong ideal.

Proof. Let \( a \in K(X) \) \( \setminus I \). Then \( 0 \ast (0 \ast a) = 0 \ast 0 = 0 \) in \( I \) but \( a \notin I \).

Example 3.14
Let \( X = (X; \ast, \circ, 0) \) be the pseudo-BCH-algebra given in Example 2.4. \( X \) has six strong ideals, namely: \( I = \{0, a, b, c\} \), \( I \cup \{d\} \), \( I \cup \{e\} \), \( I \cup \{f\} \), \( I \cup \{g, h\} \), \( X \). In \( X \), \( \{0\} \) is not a strong ideal by Proposition 3.13

In [16], K.H. Kim and E.H. Roh introduced the notion of H-ideal in BCH-algebras. Similarly, we define horizontal ideals in pseudo-BCH-algebras.

Let \( I \in \text{Id}(X) \). We say that \( I \) is a horizontal ideal of \( X \) if \( I \cap K(X) = \{0\} \). Obviously, \( \{0\} \) is a horizontal ideal of \( X \).

Remark
In pseudo-BCI-algebras, horizontal ideals were considered by G. Dymek in [4].
**Example 3.15**
Let $\mathfrak{M}$ and $I$ be given as in Example 3.3. It is not difficult to verify that $I$ is a horizontal ideal of $\mathfrak{M}$.

**Proposition 3.16**
If $\mathfrak{X}$ is a pseudo-BCH-algebra, then $K(\mathfrak{X}) = \{0\}$ if and only if every ideal of $\mathfrak{X}$ is horizontal.

**Proof.** The proof is straightforward.

**Theorem 3.17**
Let $I$ be a closed ideal of $\mathfrak{X}$. Then $I$ is horizontal if and only if $I \subseteq \text{Cen} \mathfrak{X}$.

**Proof.** Let $I$ be a closed horizontal ideal and $x \in I$. By Proposition 2.11, $0 \ast (0 \ast x) \in I$. Since $I$ is a closed ideal, from Proposition 3.1 it follows that $I$ is a subalgebra of $\mathfrak{X}$. Then

$$x \ast (0 \ast (0 \ast x)) \in I.$$  \hspace{1cm} (2)

Observe that $x \ast (0 \ast (0 \ast x)) \in K(\mathfrak{X})$. By (a5) and (a4), $0 \ast [x \ast (0 \ast (0 \ast x))] = (0 \ast x) \ast (0 \ast (0 \ast x)) = (0 \ast x) \ast (0 \ast x) = 0$, and hence

$$x \ast (0 \ast (0 \ast x)) \in K(\mathfrak{X}).$$ \hspace{1cm} (3)

From (2) and (3) it follows that $x \ast (0 \ast (0 \ast x)) \in I \cap K(\mathfrak{X}) = \{0\}$. Therefore $x \ast (0 \ast (0 \ast x)) = 0$, that is, $x \leq 0 \ast (0 \ast x)$. By (a3) and (a1) we have $0 \ast (0 \ast x) = 0 \ast (0 \ast x) \leq x$. Thus $x = 0 \ast (0 \ast x)$. Consequently, $x \in \text{Cen} \mathfrak{X}$.

Conversely, let $I \subseteq \text{Cen} \mathfrak{X}$. Then $I \cap K(\mathfrak{X}) \subseteq \text{Cen} \mathfrak{X} \cap K(\mathfrak{X}) = \{0\}$ (see (1)). From this $I \cap K(\mathfrak{X}) = \{0\}$, so $I$ is a horizontal ideal.

**Corollary 3.18**
If $\text{Cen} \mathfrak{X}$ is an ideal of $\mathfrak{X}$, then it is horizontal.

**Proof.** Let $\text{Cen} \mathfrak{X}$ be an ideal of $\mathfrak{X}$. Since $\text{Cen} \mathfrak{X}$ is a subalgebra of $\mathfrak{X}$ (see Proposition 2.7), $\text{Cen} \mathfrak{X}$ is closed by Proposition 3.1. From Theorem 3.17 we deduce that $\text{Cen} \mathfrak{X}$ is horizontal.

**Theorem 3.19**
Let $I$ be a closed ideal of $\mathfrak{X}$. Then the following statements are equivalent:

(i) $I$ is horizontal.
(ii) $x = (x \ast a) \ast (0 \ast a)$ for $x \in X$, $a \in I$.
(iii) For all $x \in X$, $a \in I$, $x \ast a = 0 \ast a$ implies $x = 0$.
(iv) For all $x \in K(\mathfrak{X})$, $a \in I$, $x \ast a = 0 \ast a$ implies $x = 0$.

**Proof.** Let $I$ be a horizontal ideal of $\mathfrak{X}$. From Theorem 3.17 it follows that $I \subseteq \text{Cen} \mathfrak{X}$. Let $x \in X$ and $a \in I$. By (pBCH-2) and (pBCH-1),

$$(x \ast a) \ast (0 \ast a) \circ x = ((x \ast a) \circ x) \ast (0 \ast a) = ((x \circ x) \ast a) \ast (0 \ast a) = (0 \ast a) \ast (0 \ast a) = 0,$$
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and hence
\[(x * a) * (0 * a) \leq x.\]  \hfill (4)

Using (pBCH-2) and (a1), we obtain
\[(x \bowtie ((x * a) * (0 * a))) * a = (x * a) \bowtie ((x * a) * (0 * a)) \leq 0 * a.\]  \hfill (5)

Since \(a \in I\) and \(0 * a \in I\), from (5) we see that
\[x \bowtie ((x * a) * (0 * a)) \in I.\]  \hfill (6)

Applying (a5) and Proposition 2.6 we get
\[0 * ((x * a) * (0 * a)) = (0 * (x * a)) \bowtie (0 * (0 * a)) = (a * x) \bowtie a = (a \bowtie a) * x = 0 * x.\]

Then by (a6) \[0 * ((x * a) \bowtie (0 * a))) = (0 * x) \bowtie (0 * x) = 0, and hence \(x \bowtie ((x * a) * (0 * a)) \in K(X).\) From this and (5) we have \(x \bowtie ((x * a) * (0 * a)) \in I \cap K(X) = \{0\},\) that is, \(x \bowtie ((x * a) * (0 * a)) = 0.\) Therefore
\[x \leq (x * a) * (0 * a).\]  \hfill (7)

Using (1), (7) and (pBCH-3) we obtain \(x = (x * a) * (0 * a).\)

\[\text{(ii) } \implies \text{ (iii) } \implies \text{ (iv) } \implies \text{ (i).} \]

Let \(x \in X, a \in I,\) and \(x * a = 0 * a.\) Then \(x = (x * a) * (0 * a) = (x * a) \cdot (x * a) = 0.\)

Therefore \(x \leq (x * a) * (0 * a).\)

We also have theorem analogous to Theorem 3.19.

**Theorem 3.20**

Let \(I\) be a closed ideal of \(X.\) Then the following statements are equivalent:

(i) \(I\) is horizontal.

(ii) \(x \bowtie (0 \bowtie a)\) for \(x \in X, a \in I.\)

(iii) For all \(x \in X, a \in I, x \bowtie a = 0 \bowtie a\) implies \(x = 0.\)

(iv) For all \(x \in K(X), a \in I, x \bowtie a = 0 \bowtie a\) implies \(x = 0.\)

**Proposition 3.21**

Let \(X\) be a pseudo-BCH-algebra. Then:

(i) If \(X\) satisfies the condition (pBCK) then the only \(\{0\}\) is a horizontal ideal of \(X\) and the only \(X\) is a strong ideal of \(X.\)

(ii) If \(0 * x = x\) for all \(x \in X,\) then every ideal of \(X\) is both strong and horizontal.

**Proof.** The proof is straightforward.

**Corollary 3.22**

If \(X\) is a pseudo-BCK-algebra, then the only \(\{0\}\) is a horizontal ideal of \(X\) and the only \(X\) is a strong ideal of \(X.\)
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