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On the differential first-order invariants for the non-splitting subgroups of the generalized Poincare group $P(1,4)$

Dedicated to Professor Andrzej Zajtz on his seventieth birthday

Abstract. The functional bases of the differential first-order invariants for all non-splitting subgroups of the generalized Poincare group $P(1,4)$ are constructed. Some applications of the results obtained are considered.

Introduction

The development of the theoretical physics has required various extensions of the four-dimensional Minkowski space and, correspondingly, various extensions of the Poincaré group $P(1,3)$. The natural extension of this group is the generalized Poincaré group $P(1,4)$. The group $P(1,4)$ is the group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. This group has many applications in theoretical and mathematical physics [1-3]. The group $P(1,4)$ has many subgroups used in theoretical physics [4-8]. Among these subgroups there are the Poincaré group $P(1,3)$ and the extended Galilei group $\tilde{G}(1,3)$ (see also [9]). Thus, the results obtained with the help of the subgroup structure of the group $P(1,4)$ will be useful in relativistic and non-relativistic physics.

The papers [10-12] are devoted to the construction of the first-order differential invariants for the splitting subgroups [4, 5, 7] of the generalized Poincaré group $P(1,4)$.

The present paper is devoted to the construction of functional bases of the differential first-order invariants for the non-splitting subgroups [4, 6-8] of the group $P(1,4)$.

Our paper is organized as follows. In the first section we introduce some notation and results concerning the Lie algebra of the group $P(1,4)$ which we use in the following. Sections 2 and 3 present our main results.

1. The Lie algebra of the group $P(1,4)$ and its non-conjugate subalgebras

The Lie algebra of the group $P(1,4)$ is given by the 15 basis elements

$$M_{\mu\nu} = -M_{\nu\mu}, \quad \mu, \nu = 0, 1, 2, 3, 4,$$

and $P_{\mu} (\mu = 0, 1, 2, 3, 4)$, satisfying the commutation relations

$$[P_{\mu}, P_{\nu}] = 0,$$

$$[M'_{\mu\nu}, P_{\sigma}] = g_{\mu\sigma}P_{\nu} - g_{\nu\sigma}P_{\mu},$$

$$[M'_{\mu\nu}, M'_{\rho\sigma}] = g_{\mu\rho}M'_{\nu\sigma} + g_{\nu\rho}M'_{\mu\sigma} - g_{\mu\sigma}M'_{\nu\rho} - g_{\nu\sigma}M'_{\mu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here, and in what follows, $M'_{\mu\nu} = iM_{\mu\nu}$.

In order to study the subgroup structure of the group $P(1,4)$ we used the method proposed in [13]. Continuous subgroups of the group $P(1,4)$ have been described in [4–8].

Further we will use the following basis elements:

$$G = M'_{40}, \quad L_{1} = M'_{32}, \quad L_{2} = -M'_{31}, \quad L_{3} = M'_{21},$$

$$P_{a} = M'_{4a} - M'_{a0}, \quad C_{a} = M'_{4a} + M'_{a0}, \quad (a = 1, 2, 3),$$

$$X_{0} = \frac{1}{2}(P'_{0} - P'_{4}), \quad X_{k} = P'_{k} \quad (k = 1, 2, 3),$$

$$X_{4} = \frac{1}{2}(P'_{0} + P'_{4}).$$

2. The differential first-order invariants of the non-splitting subgroups of the group $P(1,4)$

The group $P(1,4)$ acts on $M(1,3) \times R(u)$ (i.e., on the Cartesian product of the four-dimensional Minkowski space (of the independent variables $x_0, x_1, x_2, x_3$) and the number axis of the dependent variable $u$). The group $P(1,4)$ usually acts on $M(1,3) \times R(u)$ as a group generated by translations and rotations of this space.

Let

$$X = \sum_{i=0}^{3} \xi_{i}(x, u) \frac{\partial}{\partial x_{i}} + \eta(x, u) \frac{\partial}{\partial u}$$

be one of the basis infinitesimal operators. The first prolongation of $X$ has the form

$$X^{(1)} = X + \sum_{i=0}^{3} \left( \frac{\partial \eta}{\partial x_{i}} + \frac{\partial \eta}{\partial u} u_{i} - \sum_{j=0}^{3} u_{j} \frac{\partial \xi_{j}}{\partial x_{i}} - \sum_{j=0}^{3} u_{i} u_{j} \frac{\partial \xi_{j}}{\partial u} \right) \frac{\partial}{\partial u_{i}}.$$
Now, a function $J(x, u^{(1)})$ is a first-order differential invariant if

$$X^{(1)} \cdot J(x, u^{(1)}) = 0.$$ 

Here $u^{(1)} = (u, u_0, u_1, u_2, u_3)$ is an element of the first prolongation $R(u)^{(1)}$.

Let us consider the following representation of the Lie algebra of the group $P(1, 4)$:

\[
\begin{align*}
P_0' &= \frac{\partial}{\partial x_0}, & P_1' &= -\frac{\partial}{\partial x_1}, & P_2' &= -\frac{\partial}{\partial x_2}, & P_3' &= -\frac{\partial}{\partial x_3}, \\
P_4' &= -\frac{\partial}{\partial u}, & M_{\mu\nu}' &= -(x_\mu P_\nu' - x_\nu P_\mu'), & x_4 &\equiv u.
\end{align*}
\]

More details about this representation can be found in [14-16].

In the construction of the differential invariants it has turned out that different non-splitting subalgebras of the Lie algebra of the group $P(1, 4)$ can have the same functional basis of the first-order differential invariants. Consequently, there is no one-to-one correspondence between non-conjugate subalgebras of the Lie algebra of the group $P(1, 4)$ and corresponding differential invariants.

**Definition 1**

We call two subalgebras $L^1$ and $L^2$ of the Lie algebra of the group $P(1, 4)$ equivalent if they have the same functional basis of the first-order differential invariants.

It is possible to prove that the relation of equivalence of subalgebras $L^1$ and $L^2$ given by Definition 1 is the set-theoretical equivalence relation. With respect to this equivalence relation, all non-splitting subalgebras of the Lie algebra of the group $P(1, 4)$ split into classes of equivalent subalgebras. Each two different classes have different functional bases of the first-order differential invariants.

**Definition 2**

We call two functional bases of the first-order differential invariants of the non-splitting subalgebras of the Lie algebra of the group $P(1, 4)$ equivalent if they belong to the equivalent subalgebras.

One of the results in this section can be formulated as follows.

**Proposition**

The non-splitting subgroups of the group $P(1, 4)$ have 264 non-equivalent functional bases of the first-order differential invariants.

**Proof.** Here, we only give a sketch of the proof. Following the sketch, for the purpose of proving the Proposition, we have to use:
— the list of the non-splitting subalgebras of the Lie algebra of the group $P(1, 4)$ [17];

— the general ranks of the matrices which contain coordinates of the one time prolonged basis elements of the subalgebras of the considered Lie algebra;

— theorem on number of invariants of the Lie group of the point transformations (see, for example, [18, 19]);

— Definition 1 and Definition 2.

For all non-splitting subgroups of the group $P(1, 4)$ the functional bases of the first-order differential invariants have been constructed.

Below, for some of the non-splitting subalgebras of the Lie algebra of the group $P(1, 4)$ we give their respective basis elements and corresponding functional basis of differential invariants.

One-dimensional subalgebras

1. $(L_3 + eG + \kappa_3 X_3, e > 0, \kappa_3 < 0)$,

$$
J_1 = \left( x_0^2 - u^2 \right)^{\frac{1}{2}}, \quad J_2 = \left( x_1^2 + x_2^2 \right)^{\frac{1}{2}}, \\
J_3 = \kappa_3 \ln(x_0 + u) - ex_3, \quad J_4 = x_3 + \kappa_3 \arctan \frac{x_1}{x_2}, \\
J_5 = \frac{x_1 u_2 - x_2 u_1}{x_1 u_1 + x_2 u_2}, \quad J_6 = u_3 x_0 + u \frac{u_0 + 1}{u_0 + 1}, \\
J_7 = \frac{u_3^2}{u_0 - 1}, \quad J_8 = \frac{u_3^2 + u_3^3}{u_3^3}, \\
u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad \mu = 0, 1, 2, 3;
$$

2. $(P_3 + X_0)$,

$$
J_1 = x_1, \quad J_2 = x_2, \\
J_3 = (x_0 + u)^2 - 2x_3, \quad J_4 = x_0 - u + \frac{2}{3}(x_0 + u)^3 - 2x_3(x_0 + u), \\
J_5 = x_0 + u + \frac{u_3}{u_0 + 1}, \quad J_6 = \frac{u_1}{u_2}, \\
J_7 = \frac{u_1}{u_0 + 1}, \quad J_8 = \frac{u_3^2}{(u_0 + 1)^2} + \frac{2}{u_0 + 1}.
$$
Two-dimensional subalgebras

1. \( G, \ L_3 + dX_3, \ d < 0 \),

\[
J_1 = x_3 + d\arctan \frac{x_1}{x_2}, \quad J_2 = (x_1^2 + x_2^2)^{\frac{1}{2}},
\]

\[
J_3 = (x_0^2 - u^2)^{\frac{1}{2}}, \quad J_4 = (x_0 + u)\frac{1}{u_0 + 1},
\]

\[
J_5 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2}, \quad J_6 = \frac{u_3}{u_0 - 1},
\]

\[
J_7 = \frac{u_1^2 + u_2^2}{u_3^2};
\]

2. \( L_3 - X_4, \ P_3 \),

\[
J_1 = x_0 + u, \quad J_2 = (x_1^2 + x_2^2)^{\frac{1}{2}},
\]

\[
J_3 = x_0^2 - x_3^2 - u^2 + (x_0 + u)\arctan \frac{x_1}{x_2}, \quad J_4 = \frac{x_0 + u}{x_0 + u_0 + 1},
\]

\[
J_5 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2}, \quad J_6 = \frac{u_1^2 + u_2^2}{(u_0 + 1)^2},
\]

\[
J_7 = \frac{u_3^2}{(u_0 + 1)^2} + \frac{2}{u_0 + 1}.
\]

Three-dimensional subalgebras

1. \( G + a_1X_1 + a_3X_3, \ P_3, \ X_4, \ a_1 < 0, \ a_3 < 0 \),

\[
J_1 = x_2, \quad J_2 = x_1 - a_1 \ln(x_0 + u),
\]

\[
J_3 = x_3 - a_3 \ln(x_0 + u) + u_3\frac{x_0 + u}{u_0 + 1}, \quad J_4 = (x_0 + u)\frac{u_1}{u_0 + 1},
\]

\[
J_5 = \frac{u_1}{u_2}, \quad J_6 = \frac{u_0^2 - u_3^2 - 1}{u_1^2};
\]

2. \( L_3 - P_3 + \alpha_0X_0, \ X_3, \ X_4, \ \alpha_0 < 0 \),

\[
J_1 = (x_1^2 + x_2^2)^{\frac{1}{2}}, \quad J_2 = \alpha_0 \arctan \frac{x_1}{x_2} - x_0 - u,
\]

\[
J_3 = \arctan \frac{u_1}{u_2} = \frac{u_3}{u_0 + 1}, \quad J_4 = x_0 + u - \alpha_0\frac{u_3}{u_0 + 1},
\]

\[
J_5 = \frac{u_1^2 + u_2^2}{(u_0 + 1)^2}, \quad J_6 = \frac{u_3^2 + 2(u_0 + 1)}{(u_0 + 1)^2}.
\]
Four-dimensional subalgebras

1. \((G + a_3 X_3, \; L_3, \; P_3, \; X_4, \; a_3 < 0)\),

\[
J_1 = \left( x_1^2 + x_2^2 \right)^{\frac{3}{2}}, \quad J_2 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2},
\]
\[
J_3 = x_3 - a_3 \ln(x_0 + u) + \frac{x_0 + u}{u_0 + 1} u_3, \quad J_4 = (u_1^2 + u_2^2) \frac{(x_0 + u)^2}{(u_0 + 1)^2},
\]
\[
J_5 = \frac{u_0^2 - u_3^2 - 1}{u_1^2 + u_2^2}.
\]

2. \((L_3, \; P_1, \; P_2, \; P_3 + X_3)\),

\[
J_1 = x_0 + u, \quad J_2 = x_0^2 - x_1^2 - x_2^2 - u^2 - \frac{x_0 + u}{x_0 + u - 1} x_3^2,
\]
\[
J_3 = \frac{x_3}{x_0 + u - 1} + \frac{u_3}{u_0 + 1}, \quad J_4 = \left( \frac{x_1}{x_0 + u} + \frac{u_1}{u_0 + 1} \right)^2 + \left( \frac{x_2}{x_0 + u} + \frac{u_2}{u_0 + 1} \right)^2,
\]
\[
J_5 = \frac{u_1^2 + u_2^2 + u_3^2 + 2(u_0 + 1)}{(u_0 + 1)^2}.
\]

Five-dimensional subalgebras

1. \((G + a_2 X_1, \; P_1, \; P_2, \; P_3, \; X_4, \; a_2 < 0)\),

\[
J_1 = x_1 + \frac{x_0 + u}{u_0 + 1} u_1 - a_2 \ln(x_0 + u), \quad J_2 = x_2 + \frac{x_0 + u}{u_0 + 1} u_2,
\]
\[
J_3 = x_3 + (x_0 + u) \frac{u_3}{u_0 + 1}, \quad J_4 = (u_0^2 - u_1^2 - u_2^2 - u_3^2 - 1) \frac{(x_0 + u)^2}{(u_0 + 1)^2},
\]

2. \((L_3, \; P_1 + X_2, \; P_2 - X_1, \; X_3, \; X_4)\),

\[
J_1 = x_0 + u.
\]
\[ J_2 = \left( \frac{x_1}{x_0 + u} + \frac{u_2}{(x_0 + u)(u_0 + 1)} + \frac{u_1}{u_0 + 1} \right)^2 + \left( \frac{x_2}{x_0 + u} - \frac{u_1}{(x_0 + u)(u_0 + 1)} + \frac{u_2}{u_0 + 1} \right)^2, \]

\[ J_3 = \frac{u_3}{u_0 + 1}, \]

\[ J_4 = \frac{u_1^2 + u_2^2}{(u_0 + 1)^2} + \frac{2}{u_0 + 1}. \]

Six-dimensional subalgebras

1. \((G + a X_3, L_3 + d X_3, P_1, P_2, P_3, X_4, a < 0, d < 0)\),

\[ J_1 = (x_0 + u)^2 \left( \frac{u_1^2 + u_2^2 + u_3^2 + 2 u_0 + 2}{(u_0 + 1)^2} - 1 \right), \]

\[ J_2 = \left( x_1 + \frac{x_0 + u}{u_0 + 1} u_1 \right)^2 + \left( x_2 + \frac{x_0 + u}{u_0 + 1} u_2 \right)^2, \]

\[ J_3 = x_3 - a \ln(x_0 + u) + (x_0 + u) \frac{u_3}{u_0 + 1} + d \arctan \left( \frac{x_1(u_0 + 1) + u_1(x_0 + u)}{x_2(u_0 + 1) + u_2(x_0 + u)} \right); \]

2. \((P_1 + X_3, P_2, X_0, X_1, X_2, X_4)\),

\[ J_1 = \frac{u_1}{u_0 + 1} - x_3, \quad J_2 = \frac{u_3}{u_0 + 1}, \]

\[ J_3 = \frac{u_1^2 + u_2^2}{(u_0 + 1)^2} + \frac{2}{u_0 + 1}. \]

Seven-dimensional subalgebras

1. \((G + a_3 X_3, L_3, P_1, P_2, X_1, X_2, X_4, a_3 < 0)\),

\[ J_1 = x_3 - a_3 \ln(x_0 + u), \quad J_2 = (x_0 + u) \frac{u_3}{u_0 + 1}, \]

\[ J_3 = \frac{u_0^2 - u_1^2 - u_2^2 - 1}{u_3^2}; \]

2. \((L_3 - P_3 + a_0 X_0, P_1, P_2, X_1, X_2, X_3, X_4, a_0 < 0)\),
Eight-dimensional subalgebras

1. \((G + a_3 X_3, L_3, P_1, P_2, X_0, X_1, X_2, X_4, a_3 < 0)\),

\[ J_1 = x_3 + a_3 \ln \left( \frac{u_3}{u_0 + 1} \right), \quad J_2 = \frac{u_3^2 + u_0^2 + u_1^2 + 2(u_4 + 1)}{(u_0 + 1)^2}. \]

2. \((L_3 - X_0, P_1, P_2, P_3, X_1, X_2, X_3, X_4)\),

\[ J_1 = \frac{u_1^2 + u_2^2 + u_3^2 + 2(u_0 + 1)}{(u_0 + 1)^2}. \]

3. On some applications of the results obtained

The differential invariants of the local Lie groups of the point transformations play an important role in the group-analysis of differential equations (see, for example [18-28]). In particule, with the help of these invariants we can construct differential equations with non-trivial symmetry groups.

In our case the considered equations can be written in the following form (see, for example [18-20]):

\[ F(J_1, J_2, \ldots, J_t) = 0, \]

where \( F \) is an arbitrary smooth function of its arguments, \( \{J_1, J_2, \ldots, J_t\} \) is a functional basis of the first-order differential invariants of the non-splitting subgroups of the group \( P(1,4) \).

Since the Lie algebra of the group \( P(1,4) \) contains, as subalgebras, the Lie algebra of the Poincaré group \( P(1,3) \) and the Lie algebra of the extended Galilei group \( G(1,3) \) (see also [9]), the results obtained can be used in relativistic and non-relativistic physics.

References


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